On an Equivalence in Discrete Extremal Problems

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Abstract

We introduce some equivalence relations on graphs and posets and prove that they are closed under the cartesian product operation. These relations concern the edge-isoperimetric problem on graphs and the shadow minimization problems on posets. For a long time these problems have been considered quite independently. We present close connections between them. In particular we show that a number of known results concerning the edge-isoperimetric problem for concrete families of graphs are direct consequences of the Macauleyness of appropriate posets.

Keywords: Isoperimetric problem, Macaulay poset, compression, cartesian product.

1 Introduction

Let $G = (V_G, E_G)$ be a graph. We consider the following general problem: given a function $F : 2^{V_G} \mapsto \mathbb{R}$ and a number m $(1 \leq m \leq |V_G|)$, find an m-element subset $A \subseteq V_G$ with maximum (or minimum) value of $F(A)$ among all the m-element subsets of V_G. Such subsets are called *optimal*.

Similar problems arise in a number of practical situations. We say that optimal subsets satisfy the *nested solutions property (NS)* if there exists a total order $\mathcal{O}$ on the set V_G such that for any $t = 1, \ldots, |V_G|$, the collection of the first t vertices in this order is an optimal subset. In this case we call the order $\mathcal{O}$ the *optimal order*.

We concentrate on the graphs representable as cartesian products. Given two graphs $G_1 = (V_{G_1}, E_{G_1})$ and $G_2 = (V_{G_2}, E_{G_2})$, their *cartesian product* $G_1 \times G_2$ is defined as a graph with the vertex set $V_{G_1} \times V_{G_2}$ and the edge set

$$\{((x, y), (u, v)) \mid x = u \text{ and } (y, v) \in E_{G_2}, \text{ or } (x, u) \in E_{G_1} \text{ and } y = v\}.$$

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Now let $(P^{(1)}, \subseteq_{P^{(1)}})$ and $(P^{(2)}, \subseteq_{P^{(2)}})$ be posets. We define the cartesian product of these posets as a poset with the element set $P^{(1)} \times P^{(2)}$ and with the partial order $\subseteq_\times$ defined as follows: $(x_1, y_1) \subseteq_\times (x_2, y_2)$ iff $x_1 \subseteq_{P^{(1)}} x_2$ and $y_1 \subseteq_{P^{(2)}} y_2$. Since the cartesian product is an associative operation, the products of more than two graphs or posets are well defined.

A lot of extremal problems for the cartesian product of graphs and posets have been considered in the literature. Practically in all cases solutions of these problems satisfy the NS property. One of the main questions we investigate in this paper is how the NS property of the optimal subsets in graphs (or posets) can be used to construct optimal subsets in cartesian products of these graphs (resp. posets). In Section 2 we introduce three extremal problems we deal with in the paper.

Section 3 is devoted to an equivalence relation on graphs. We apply this relation to the edge-isoperimetric problem (shortly EIP) and derive a solution of this problem for the cartesian products of arbitrary trees.

In the next two sections we introduce an equivalence relation on posets. Thus, in Section 4 we consider the problem of constructing maximum weight ideals in posets (the MWI problem), and present relations between this problem and the EIP problem in graphs.

Section 5 deals with the shadow minimization problem on posets. We introduce Macaulay posets and show their applicability to the MWI problem and, thus, to the EIP on related graphs.

In Section 6 we present some examples where our approach works well and conclude the paper with final remarks in Section 7.

2 Three extremal problems

2.1 Edge Isoperimetric Problem on graphs (EIP)

Let $G = (V_G, E_G)$ be a graph and $A \subseteq V_G$. Denote

$$\begin{aligned} E_G(A) &= |\{(v,w) \in E(G) \mid v, w \in A\}|, \\ E_G(m) &= \max_{|A|=m} E_G(A), \\ \delta_G(m) &= E_G(m) - E_G(m-1), \quad \delta_G(1) = 0. \end{aligned}$$

The EIP problem can be formulated as follows: for a given m, $1 \le m \le |V_G|$, find a subset $A^* \subseteq V_G$ such that $|A^*| = m$ and $E_G(A^*) = E_G(m)$.

The EIP is considered, for example, for the n-cube [10], for the cartesian product of complete graphs [16] and for the cartesian product of chains [7] (see also [1]). In all these cases the optimal subsets are nested both for the original graphs and their cartesian products. For more information concerning the EIP and its applications readers are referred to the survey [5].

2.2 Maximum Weight Ideals in posets (MWI)

Let $(P, \subseteq_P)$ be a finite poset. The poset $(P, \subseteq_P)$ is called *ranked* if there exists a function $r_P : P \mapsto \mathbb{N}$ such that $\min_{x \in P} r_P(x) = 0$ and $r_P(x) + 1 = r_P(y)$ whenever $x \subseteq_P y$ and there is no z with $x \subseteq_P z \subseteq_P y$. We call the numbers $r_P(x)$ and $r_P = \max_{x \in P} r_P(x)$ the *rank* of x and P respectively. The set

$$P_i = \{x \in P \mid r_P(x) = i\}$$

is called the i^{th} *level* of P. It can be shown that the cartesian product of ranked posets is a ranked poset too.

A subset $I \subseteq P$ is called *ideal* if the conditions $x \in I$ and $y \subseteq x$ imply $y \in I$. Let $w_P : P \mapsto \mathbb{R}^+$ be a *weight function*. The weight function w_P is called *rank-symmetric* if $w_P(x) = w_P(y)$ whenever $r_P(x) = r_P(y)$. In this case we shall represent w_P as a tuple $(w_0, w_1, \ldots, w_{r(P)})$ with w_i being the weight of elements of P_i.

Let $I \subseteq P$ be an ideal. Denote

$$\begin{aligned} W_P(I) &= \sum_{x \in I} w_P(x), \qquad (1) \\ W_P(m) &= \max_{|I|=m} W_P(I), \\ \delta_P(m) &= W_P(m) - W_P(m-1), \quad \delta_P(1) = 0, \end{aligned}$$

with the maximum running over the ideals of P. Consider the MWI problem: for a fixed m, $1 \le m \le |P|$, find an ideal $I^* \subseteq P$ such that $|I^*| = m$ and $W_P(I^*) = W_P(m)$.

A lot of results concerning the MWI problem for various posets and weight functions can be found in [2,3,6,8,9]. In general, the MWI problem with a rank-symmetric weight function is very close to the shadow minimization problem (SMP) that we consider in the next subsection. Solution of the MWI problem is known for posets where the optimal subsets with respect to the SMP satisfy the NS property.

Let $(P, \subseteq_P)$ be a ranked poset. For a subset $A \subseteq P_i$ and $i > 0$ define the *shadow* of A as

$$\Delta(A) = \{x \in P_{i-1} \mid x \subseteq_P y \text{ for some } y \in A\}$$

We set $\Delta(A) = \emptyset$ for any $A \subseteq P_0$. The *shadow minimization problem* (SMP): for fixed $i > 0$ and m, $1 \leq m \leq |P_i|$, find a subset $A \subseteq P_i$ such that $|A| = m$ and $|\Delta(A)| \leq |\Delta(B)|$ for any $B \subseteq P_i$ with $|B| = m$.

A classical result in this area is the well-known theorem proved by Kruskal [13] and Katona [12] for the n-cube. Clements and Lindström [8] extended this result to the *lattice of multisets* (i.e. the cartesian product of chains), and in [3,14,15] the SMP problem is solved for the cartesian product of stars. In all mentioned cases the optimal subsets satisfy the NS property. A short survey on the SMP and a lot of its applications can be found in [9].

3 An equivalence in the EIP

For some graphs $G_1, \ldots, G_n$ suppose that the optimal subsets with respect to the EIP satisfy the NS property. For $i = 1, \ldots, n$ we assume that $V_{G_i} = \{1, \ldots, |V_{G_i}|\}$, and for any $m = 1, \ldots, |V_{G_i}|$ assume that the set $\{1, \ldots, m\}$ is optimal.

Let $G = G_1 \times \cdots \times G_n$ and consider the EIP for G. Fix some i, $1 \leq i \leq n$, and denote

$$G_i^{\perp} = G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n.$$

Now for $v = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \in G_i^{\perp}$ denote by $G_i(v)$ the subgraph of G induced by the vertex set $\{(w_1, \ldots, w_n) \in V_G \mid w_j = v_j,\ j \neq i\}$. Obviously, each subgraph $G_i(v)$ is isomorphic to G_i.

Let $A \subseteq V_G$ and denote $A_i(v) = A \cap V_{G_i(v)}$. Let $C_i(A_i(v))$ be the optimal subset of $V_{G_i(v)}$ that is isomorphic to $\{1, \ldots, |A_i|\} \subseteq V_{G_i}$. Consider the set $C_i(A) = \bigcup_v A_i(v)$ with the union running over all $v \in V_{G_i^{\perp}}$.

We claim $E_G(C_i(A)) \geq E_G(A)$. Indeed,

$$E_G(A) \leq \sum_v E_{G_i(v)}(|A_i(v)|) + \sum_{(u,v)} \max\{|A_i(u)|,\, |A_i(v)|\}$$

with the sums running over $v \in V_{G_i^{\perp}}$ and $(u, v) \in E_{G_i^{\perp}}$ respectively. It can be easily shown that for the set $C_i(A)$ this inequality is strict. Hence, applying the operations C_i for $i = 1, \ldots, n$ sufficiently many times results in an optimal set A^* such that $C_i(A^*) = A^*$ for $i = 1, \ldots, n$. We call such a set *compressed*.

Taking into account $E_{G_i}(\{1, \ldots, m\}) = \sum_{k=1}^{m} \delta_{G_i}(k)$ for $i = 1, \ldots, n$, it can be easily shown that for the compressed set A^* one has

$$E_G(A) = \sum_{(v_1, \ldots, v_n) \in A^*} \delta_{G_i}(v_i). \tag{2}$$

Now consider the poset $(Q, \leq_\times)$ with $Q = V_{G_1} \times \cdots \times V_{G_n}$ and the partial order $\leq_\times$ defined as follows: $(x_1, \ldots, x_n) \in Q$ is smaller than $(y_1, \ldots, y_n) \in Q$ in order $\leq_\times$ iff $x_i \leq y_i$ for $i = 1, \ldots, n$. Obviously, $(Q, \leq_\times)$ is isomorphic to a lattice of multisets.

Since the set V_G and Q are the same, the partial order $\leq_\times$ on Q provides a bijection between the compressed subsets of V_G and the ideals of Q. Denote by $I_Q(A^*)$ the ideal of Q that corresponds to the compressed set $A^* \subseteq V_G$. We assign a weight with an element $(v_1, \ldots, v_n) \in Q$ given by

$$w_Q(v_1, \ldots, v_n) = \sum_{i=1}^{n} \delta_{G_i}(v_i) \tag{3}$$

It follows from (1) - (3) that

$$E_G(A^*) = W_Q(I_Q(A^*)).$$

Thus, we have the following lemma:

Lemma 1 *Let $G_1, \ldots, G_n$ be graphs and for each G_i suppose that the optimal subsets with respect to the EIP satisfy the NS property. Then for the EIP on the graph $G_1 \times \cdots \times G_n$ and the MWI problem on the poset $(Q, \preceq)$ with the weight function (3) it holds*

$$E_{G_1 \times \cdots \times G_n}(m) = W_Q(m) \tag{4}$$

for any $m \geq 1$.

Let G and H be graphs with $|V_G| = |V_H|$ and let the optimal subsets of V_G and V_H with respect to the EIP satisfy the NS property. We say that G and H are *E-equivalent*, if $E_G(m) = E_H(m)$ for $m = 1, \ldots, |V_G|$. Lemma 1 implies the following result:

Theorem 2 *Let $\{G_i, H_i\}$, $i = 1, \ldots, n$, be a set of pairwise E-equivalent graphs. Then*

$$E_{G_1 \times \cdots \times G_n}(m) = E_{H_1 \times \cdots \times H_n}(m)$$

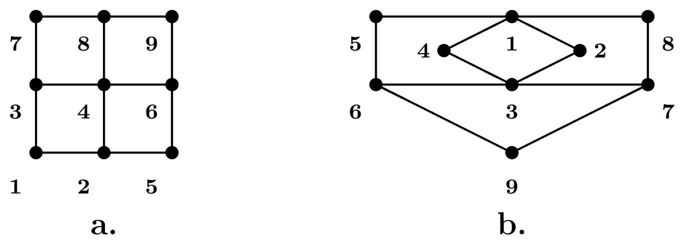

Fig. 1. Example of E-equivalent graphs

for any $m \geq 1$. Moreover, the optimal subsets of $G_1 \times \cdots \times G_n$ satisfy the NS property iff it is so for $H_1 \times \cdots \times H_n$.

For example, consider the graphs shown in Fig. 1a and Fig. 1b. The optimal subsets in these graphs satisfy the NS property. The corresponding optimal orders are shown in the figure. By using these orders it can be easily shown that the graphs are E-equivalent. Hence, by Theorem 2, a solution for the EIP for the cartesian products of chains [1,7] (and, thus, for grids in Fig. 1a) implies a solution for the EIP for the cartesian products of graphs in Fig. 1b.

Therefore, in order to solve the EIP on cartesian products of graphs, it is sufficient solve this problem for the products of simplest graphs from the corresponding equivalence classes. As an application of this principle let us consider the EIP for trees with p vertices. It is obvious that any such a tree is E-equivalent to the chain with p vertices. Thus, we have the following result:

Corollary 3 *Let T_i be a tree with p_i vertices $(i = 1, \ldots, n)$. Then the optimal subsets with respect to the EIP for $G = T_1 \times \cdots \times T_n$ satisfy the NS property. Moreover, $E_G(m) = E_Q(m)$ for any $m \geq 1$, with Q being the cartesian product of n chains with $p_1, \ldots, p_n$ vertices respectively.*

As an example, consider the cartesian product of two chains Z with 4 vertices each. The optimal orders on V_Z and $V_{Z \times Z}$ (cf. [1,7]) are shown in Fig. 2a and Fig. 2c respectively. Now consider the tree T shown in Fig. 2d and its optimal order, that induces a labeling of $T \times T$ (cf. Fig. 2e). Taking the vertices of $T \times T$ in the same order as the corresponding vertices of $Z \times Z$ (see Fig. 2c) results in an optimal order for $T \times T$ shown in Fig. 2f.

4 From the EIP on graphs to the MWI in posets

In this section we show that the EIP for a given graph is equivalent to the MWI problem for some related poset with a rank-symmetric weight function. We start with an equivalence principle for the MWI problem.

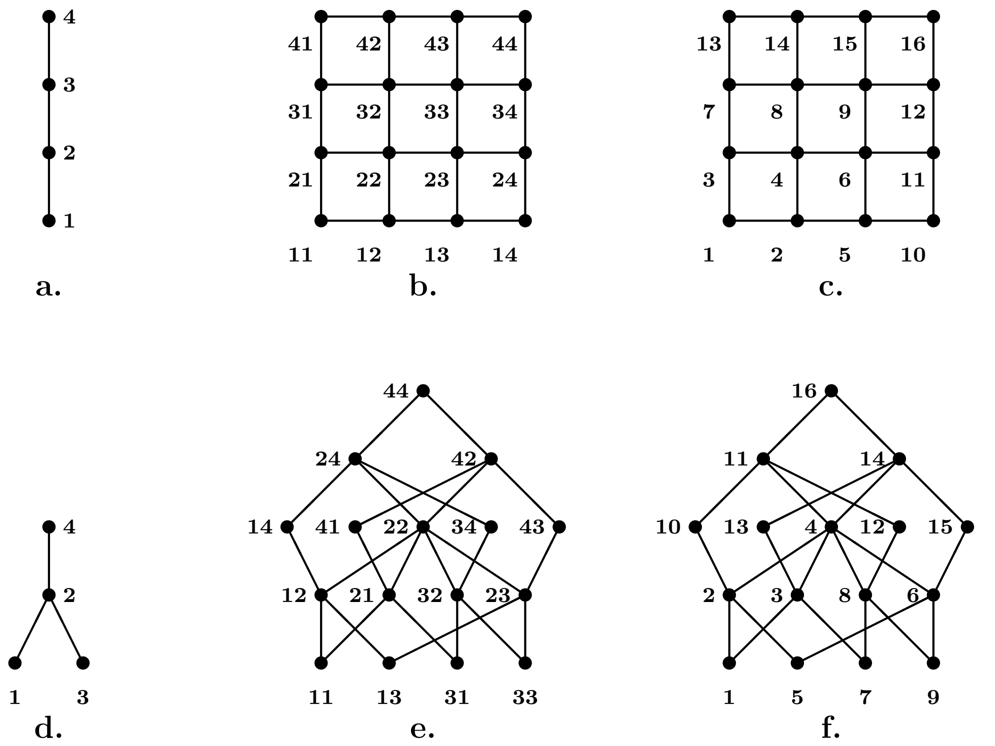

Fig. 2. The EIP for products of trees

Let $(P^{(i)}, \subseteq_{P^{(i)}})$, $i = 1, \ldots, n$, be ranked posets with weight functions $w_{P^{(i)}}$. For each poset $(P^{(i)}, \subseteq_{P^{(i)}})$ suppose that the optimal ideals with respect to the MWI problem satisfy the NS property. Furthermore, let $P^{(i)} = \{1, \ldots, |P^{(i)}|\}$, $i = 1, \ldots, n$, and for any $p = 1, \ldots, |P^{(i)}|$ assume that the set $\{1, \ldots, p\}$ is an optimal ideal.

Denote $P = P^{(1)} \times \cdots \times P^{(n)}$ and consider the MWI problem on P with the weight function being defined as

$$w_P(v_1, \ldots, v_n) = \sum_{i=1}^{n} w_{P^{(i)}}(v_i). \tag{5}$$

Let us again introduce the poset $(Q, \leq_\times)$ with $Q = P$ and the partial order $\leq_\times$ as in Section 3, and assign a weight with each element $(v_1, \ldots, v_n) \in Q$ given by

$$w_Q(v_1, \ldots, v_n) = \sum_{i=1}^{n} \delta_{P^{(i)}}(v_i). \tag{6}$$

Lemma 4 *Let $(P^{(i)}, \subseteq_{P^{(i)}})$ $(i = 1, \ldots, n)$ be posets and for each of then suppose that the optimal ideals with respect to the MWI problem satisfy the NS property. Then for the MWI problems on the posets $(P^{(1)} \times \cdots \times P^{(n)}, \subseteq_\times)$*

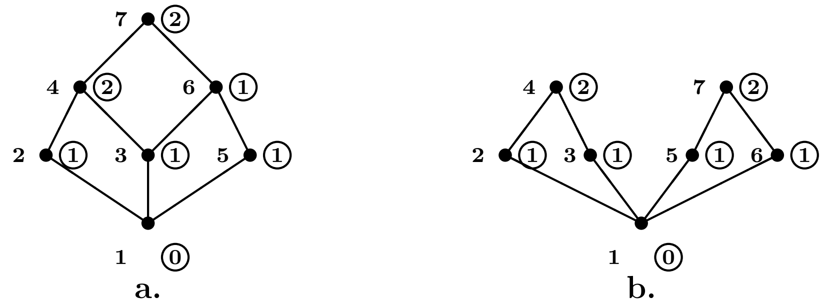

Fig. 3. Example of W-equivalent posets

and $(Q, \leq_\times)$ with the weight functions (5) and (6) respectively it holds

$$W_{P^{(1)}\times\cdots\times P^{(n)}}(m) = W_Q(m) \tag{7}$$

for any $m \geq 1$.

Proof. We just sketch the proof because it is quite similar to the proof of Lemma 1. Denote $P = P^{(1)} \times \cdots \times P^{(n)}$ and let

$$P_\perp^{(i)} = P^{(1)} \times \cdots \times P^{(i-1)} \times P^{(i+1)} \times \cdots \times P^{(n)}.$$

Now for $v = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n) \in P_\perp^{(i)}$ consider the subposet of $(P, \subseteq_\times)$ with the element set $P^{(i)}(v) = \{(w_1, \ldots, w_n) \in P \mid w_j = v_j,\ j \neq i\}$ and the induced partial order. Obviously, such a subposet is isomorphic to $P^{(i)}$. For an ideal $I \subseteq P$ denote $I^{(i)}(v) = I \cap P^{(i)}(v)$.

Now fix i and replace for each $v \in P_\perp^{(i)}$ the set $I^{(i)}(v)$ with the set which is isomorphic to $\{1, \ldots, |I^{(i)}(v)|\} \subseteq P^{(i)}$. This transforms the ideal I into some ideal $I' \subseteq P$ with $W_P(I') \geq W_P(I)$. After a finite number of such transformations for $i = 1, \ldots, n$ one gets a compressed ideal I^*. Similarly to the proof of Lemma 1 there exists a bijection between compressed ideals in P and ideals in Q. Thus, the ideal $I^* \in P$ corresponds to some ideal $\tilde{I} \in Q$. Moreover, (5) and (6) imply $W_P(I^*) = W_Q(\tilde{I})$, and the lemma follows. □

Let $(P, \subseteq_P)$ and $(R, \subseteq_R)$ be posets with $|P| = |R|$, and let the optimal ideals in each of them satisfy the NS property. We say that $(P, \subseteq_P)$ and $(R, \subseteq_R)$ are *W-equivalent*, if $W_P(m) = W_R(m)$ for $m = 1, \ldots, |P|$.

For example, the posets shown in Fig. 3a and 3b are W-equivalent. The numbers in circles represent the weights and the non circled numbers represent the optimal orders.

Lemma 4 implies the following theorem:

Theorem 5 *Let $\{(P^{(i)}, \subseteq_{P^{(i)}}), (R^{(i)}, \subseteq_{R^{(i)}})\}$, $i = 1, \ldots, n$, be a set of pairwise W-equivalent posets. Then*

$$W_{P^{(1)} \times \cdots \times P^{(n)}}(m) = W_{R^{(1)} \times \cdots \times R^{(n)}}(m)$$

for any $m \geq 1$, with the weight function for the cartesian product being defined according to (5). Moreover, the optimal ideals of $P^{(1)} \times \cdots \times P^{(n)}$ satisfy the NS property iff it is so for $R^{(1)} \times \cdots \times R^{(n)}$.

It follows from Fig. 3 that in some cases the MWI problem on a poset with a rather complicated weight function is equivalent to same problem on some W-equivalent poset with a rank-symmetric weight function. It is important, because the presently known techniques to solve the MWI problem is applicable to posets with rank-symmetric weight functions only (cf. Section 5).

As we have seen in Section 3, the EIP problem for products of even simple graphs, such as a chain, leads to the MWI problem for a lattice of multisets with a non rank-symmetric weight function in general. Now we present a way for direct replacing the EIP problem on a graph with the MWI problem for some appropriate poset with a rank-symmetric weight function.

Let G be a graph and let the optimal subsets of V_G with respect to the EIP satisfy the NS property. We say that a graph G is *represented* by a ranked poset $(P, \subseteq_P)$ with $|P| = |V_G|$ if the optimal ideals of P with respect to the MWI problem and the weight function

$$w_P(v) = r_P(v), \quad v \in P, \tag{8}$$

satisfy the NS property, and

$$\delta_G(m) = \delta_P(m), \quad m = 1, \ldots, |V_G|. \tag{9}$$

For example, the Petersen graph (see Fig. 4a) is represented by the poset shown in Fig. 4b (without dotted lines) with the rank-symmetric weight function shown in the circles in Fig. 4b.

Theorem 6 *Let G_i $(i = 1, \ldots, n)$ be graphs and for each i suppose that G_i is represented by a poset $(P^{(i)}, \subseteq_{P^{(i)}})$. Then for the poset $(P, \subseteq_\times) = (P^{(1)} \times \cdots \times P^{(n)}, \subseteq_\times)$ with the weight function (8) it holds*

$$E_{G_1 \times \cdots \times G_n}(m) = W_{P^{(1)} \times \cdots \times P^{(n)}}(m) \tag{10}$$

for any $m \geq 1$. Moreover, the optimal subsets in $G_1 \times \cdots \times G_n$ satisfy the NS property iff it is so for the optimal ideals of P.

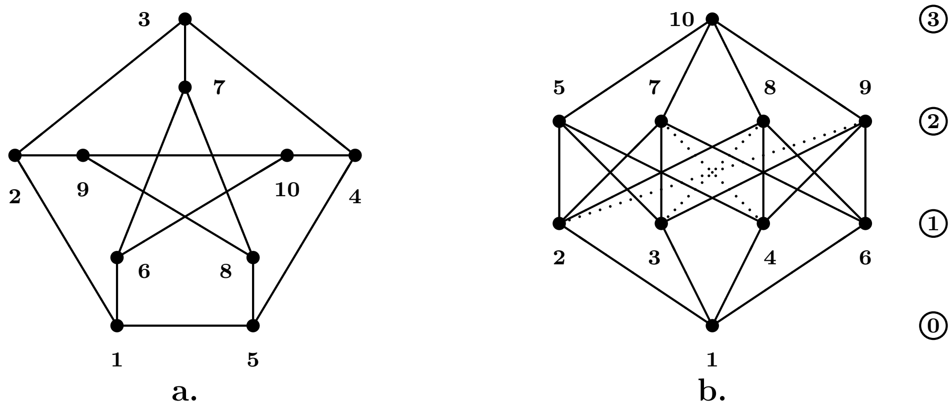

Fig. 4. The Petersen graph and its representing poset

Proof. Let $v = (v_1, \dots, v_n) \in P$. It follows from (8) that $w_{P^{(i)}}(v) = r_{P^{(i)}}(v)$ for $i = 1, \dots, n$. Since $r_P(v) = \sum_{i=1}^{n} r_{P^{(i)}}(v_i)$ and since $w_P(v) = \sum_{i=1}^{n} w_{P^{(i)}}(v_i)$ (cf. (5)), then $r_P(v) = w_P(v)$.

By Lemmas 1 and 4 the EIP for $G_1 \times \cdots \times G_n$ and the MWI problem for $(P, \subseteq_\times)$ are equivalent to the MWI problems for the poset $(Q, \leq_\times)$ with the weight functions w'_Q and w''_Q being defined according to (3) and (6) respectively. Now (9) implies $\delta_{G_i}(m) = \delta_{P^{(i)}}(m)$ for any m and $i = 1, \dots, n$. Hence, $w'_Q = w''_Q$ and (10) follows from (4) and (7). □

Theorem 7 *Let G be a graph and let the optimal subsets of V_G with respect to EIP satisfy the NS property. Then G is represented by some ranked poset.*

Proof. We use induction on $|V_G|$. For $|V_G| = 1$ the representing poset is trivial.

For $|V_G| > 1$ let $V_G = \{1, \dots, |V_G|\}$ and for each $p = 1, \dots, |V_G|$ assume that the subsets $\{1, \dots, p\} \subseteq V_G$ are optimal. Note that for $p < |V_G|$ these subset are also optimal for the subgraph G' which is induced by the vertex set $\{1, \dots, |V_G| - 1\}$. Construct the representing poset $(P', \subseteq_{P'})$ for G' by induction. Now extend P' by adding a new element v at level $\delta_G(|V_G|)$ and extend the partial order $\subseteq_{P'}$ by setting v to be greater than any element of P' at level $\delta_G(|V_G|) - 1$. This procedure results in a poset $(P, \subseteq_P)$.

The correctness of this construction is provided by the following simple facts: $\delta_G(i) \leq \delta_G(i-1) + 1$ for $i = 1, \dots, |V_G|$ and, thus, for any integer x with $1 \leq x \leq \max_j \delta_G(j)$ there exists an i with $\delta_G(i) = x$. Moreover, since G is connected, then $r_P(v) \geq 1$. Therefore, the poset $(P, \subseteq_P)$ is ranked.

In order to complete the proof it suffices to show that the optimal ideals of P satisfy the NS property and that the element v is the largest one in some

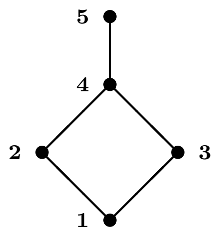

Fig. 5. A poset that represents no graph

optimal order on P. For this consider an ideal $I \subseteq P$. Assuming $|I| < |P|$, we prove that there exists an ideal $I' \subseteq P'$ with $W_P(I') \geq W_P(I)$.

Indeed, if $I \not\subseteq P'$ then $v \in I$. Denote $I'' = I \setminus \{v\}$. Then $I'' \subseteq P'$ is an ideal. Note that

$$\{x \in P' \mid r_P(x) < r_P(v)\} \subseteq I''. \tag{11}$$

Obviously, I'' can be extended to an ideal $I' \subseteq P'$ by adding to it some element $u \in P' \setminus I''$. Now (11) implies $r_P(u) \geq r_P(v)$. Therefore, $(P, \subseteq_P)$ represents G and we have the theorem. $\square$

The representing poset for the Petersen graph (cf. Fig. 4a), which is constructed by this method, differs from the one shown in Fig. 4b in the dotted lines. This example shows that, although the element set of the representing poset and the rank of each element are defined uniquely by G, the partial order is not uniquely defined in general.

It is interesting that not any poset represents some graph. Consider, for example, the poset shown in Fig. 5 together with an optimal order. If the corresponding graph G exists, then $\delta_G(i)$ for $i = 1, \ldots, 5$ have to be $0, 1, 1, 2, 3$ respectively. Hence, the subgraph of G induced by the first four vertices is a 4-cycle and the fifth vertex has degree 3. However, such a graph necessarily contains a 3-cycle. Thus, the three first values of $\delta_G(i)$ should be $0, 1, 2$.

Therefore, the EIP in graphs is equivalent to the MWI problem for an appropriate poset and the last problem is in a sense more general. However, there is a powerful approach to solve the MWI problem, which we consider in the next section.

5 Macaulay posets and MWI problem

Let $(P \subseteq_P)$ be a ranked poset and let $\preceq$ be a total order on P. For $z \in P_i$ denote

$$\mathcal{F}_i(z) = \{x \in P_i \mid x \preceq z\}.$$

The poset $(P, \subseteq_P)$ is called *Macaulay*, if there exists a total order $\preceq$ (called *Macaulay order*) that satisfies the following properties:

N$_1$ (*nestedness*) : For any $z \in |P_i|$, and any $i > 0$ it holds $|\Delta(\mathcal{F}_i(z))| \leq |\Delta(A)|$ for any $A \subseteq P_i$ with $|A| = |\mathcal{F}_i(z)|$;
N$_2$ (*continuity*) : For $i > 0$ it holds $\Delta(\mathcal{F}_i(z)) = \mathcal{F}_{i-1}(z')$ for some $z' \in P_{i-1}$.

Examples of Macaulay posets include the n-cube (cf. the Kruskal-Katona theorem [12,13]) and the cartesian product of chains (cf. the Clements-Lindström theorem [8]). For more information on Macaulay posets readers are referred to [9].

Let $(P, \subseteq_P)$ be a Macaulay poset with a rank-symmetric weight function w_i such that

$$w_0 \leq w_1 \leq \ldots \leq w_{r(P)}.$$

We call such a function *monotone*. Let $A \subseteq P$ and denote $A_i = A \cap P_i$ for $i = 0, \ldots, r(P)$.

We construct a new total order $\mathcal{O}^*$ on the set P as follows. Set the first element of P in order $\mathcal{O}^*$ to be the first element of P_0 in the Macaulay order $\preceq$. Assume that $l \geq 1$ elements of P have been ordered and denote by A the collection of them. Consider the set $B = \{a \in P \setminus A \mid \Delta(a) \subseteq A\}$. Note that $B \neq \emptyset$ for any $l < |P|$, since $\Delta(a) = \emptyset$ for any $a \in P_0$. Let $C \subseteq B$ be the elements of B of maximal rank, and let $c \in C$ be the smallest element of C in the Macaulay order $\preceq$. We set the element c to be the $(l+1)^{st}$ element in order $\mathcal{O}^*$. Denote by $\mathcal{O}^*(l)$ the initial segment of length l of the order $\mathcal{O}^*$. It is easily shown that $\mathcal{O}^*(l)$ is an ideal for any $l = 0, \ldots, |P|$.

Theorem 8 ([3,4,9]). *Let $(P, \subseteq_P)$ be a Macaulay poset with some monotone weight function. Then $W_P(I) \leq W_P(\mathcal{O}^*(|I|))$ for any ideal $I \subseteq P$.*

Since the weight function that satisfies (8) is monotone and rank-symmetric, then Theorem 8, applied to a Macaulay poset P, implies that the optimal ideals with respect to the MWI problem satisfy the NS property. If P represents some graph G, then (by Theorem 6) the optimal subsets of V_G with respect to EIP problem also satisfy the NS property and, thus, can be constructed by using the Macaulay order on P.

In the following section we demonstrate how this approach can be applied to some important graph families.

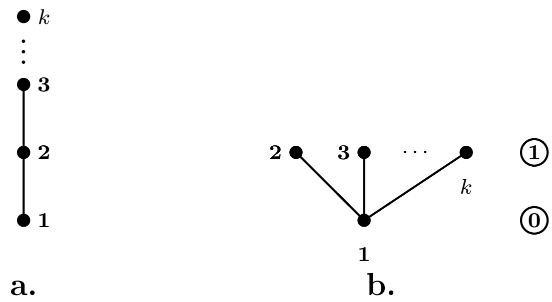

Fig. 6. A chain and a star poset

6 Some applications

6.1 Grids and the star posets

Consider the EIP for the $k \times \cdots \times k$ grid, i.e. the cartesian product of n chains P_i $(i = 1, \ldots, n)$ with k vertices each (cf. Fig. 6a).

Each P_i is represented by the star poset shown in Fig. 5b. Therefore, by Theorems 8 and 6 the solution of the EIP for the grid (see [1,7]) follows from the solution of the SMP problem for the cartesian product of n star posets [3,14,15].

6.2 The Hamming graphs and grid posets

Consider the EIP for a *Hamming graph*, i.e. the cartesian product of n complete graphs with $k_1, \ldots, k_n$ vertices respectively. We denote this graph by $H(k_1, \ldots, k_n)$. If $k_1 \geq \cdots \geq k_n \geq 2$ then the *lexicographic order* is an optimal order [16]. The lexicographic order on the set of vectors with integral entries is defined as follows: $(a_1, \ldots, a_n)$ is greater than $(b_1, \ldots, b_n)$ iff there exist an $i \geq 1$ such that $a_j = b_j$ for $j = 1, \ldots, i-1$ and $a_i > b_i$.

Obviously, the complete graph with k_i vertices is represented by the chain poset shown in Fig. 6a. The SMP for the poset represented by the cartesian product of chains has been considered in [8]. The Clements-Lindström theorem implies that the lexicographic order is the Macaulay order for this poset. Moreover, the lexicographic order provides optimal ideals with respect to MWI problem on this poset [8]. Therefore, by Theorems 8 and 6 the solution of the SMP for $H(k_1, \ldots, k_n)$ follows from the Clements-Lindström theorem and it is not surprising that the lexicographic order works well for both problems.

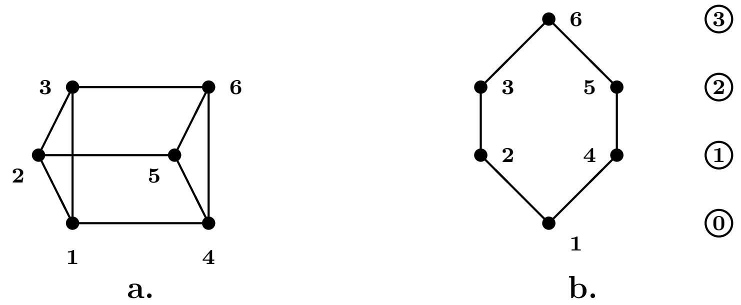

Fig. 7. A Hamming graph and a torus poset

6.3 The Hamming graphs and torus posets

Let C_{2k} be a cycle with $2k$ vertices. We consider C_{2k} as a ranked poset with one maximal and one minimal element (cf. Fig. 7b). The solution of the SMP for the cartesian product of n cycles follows from [11]. Since graph $H(2, k)$ (cf. Fig. 7a) is represented by poset C_{2k}, Theorems 8 and 6 provide a solution for the EIP on the Hamming graphs of the form $H(k_1, \ldots, k_n) \times H(\underbrace{2, \ldots, 2}_{n})$.

7 Concluding remarks

Since the MWI problem provides a tool to solve the EIP, it is reasonable to study this problem separately, but *not only* as a consequence of the SMP. It is particularly interesting to understand which properties have to be claimed from the optimal ideals with respect to the MWI problem on a poset $(P, \subseteq_P)$ in order to deduce a solution for the SMP for this poset. The only known to us research in this direction is [6], where it is shown that the SMP for the cartesian product of chains is a direct consequence of the MWI problem on this poset, and, thus, both problems for this particular poset are essentially equivalent.

Evidently, the SMP and the MWI problems are closely related but, however, are nonequivalent. Consider, for example, the graph G shown in Fig. 8a. It can be shown that the lexicographic order is optimal with respect to the EIP on $G \times G$. Graph G is represented by the poset P shown in Fig. 8b. Theorem 6 implies that optimal ideals with respect to the MWI problem for the poset $P \times P$ satisfy the NS property. However, the poset $P \times P$ is not Macaulay.

Macaulay posets have many applications in combinatorics (cf. [9]). That is why the problem of constructing Macaulay posets is very important. It is also important to find new Macaulay posets representable as cartesian products. For example, what about the cartesian products of posets shown in Fig. 4b?

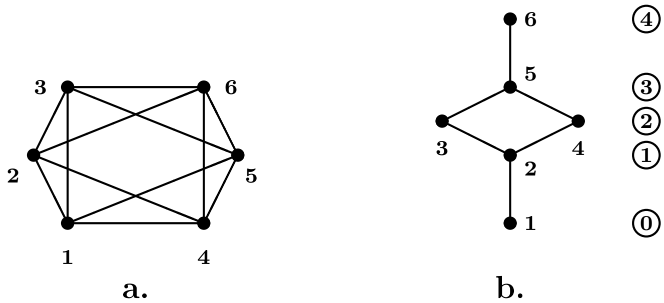

Fig. 8. A not Macaulay poset for which the MWI problem has nested solutions

We presented a number of examples of graphs where the lexicographic order provides nestedness in the EIP for the cartesian products of these graphs. It would be interesting to have a characterization of such graphs.

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