Minimization of the Shadows in the Partial Mappings Semilattice

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1 Introduction

Let g,h be partial mappings of $\{1, 2, ..., n\}$ into $\{0, 1, ..., k\}$ and $D(g)$,$D(h)$ be their domains. We say that h is greater or equal to g iff $D(h) \subseteq D(g)$ and $g(x) = h(x)$ for all $x \in D(h)$. The collection of all partial mappings with the order just defined forms the ranked poset, which we denote by F_k^n. We may assign to each mapping g a vector $\tilde{a} = (a_1, ..., a_n)$ such that $a_i \in \{-1, 0, ..., k\}$ where $a_i = g(i)$ for $i \in D(g)$ and $a_i = -1$ for $i \notin D(g)$. So, up to the end of the paper we will not make a difference between mappings and vectors.

For $\tilde{a} = (a_1, ..., a_n)$ denote

$$\begin{aligned} \|\tilde{a}\| &= |\{i : a_i = -1\}|, \\ F_{k,t}^n &= \{\tilde{a} \in F_k^n : \|\tilde{a}\| = t\}. \end{aligned}$$

Let $A \subseteq F_{k,t}^n$. We call the r-shadow of A the collection of all $\tilde{b} \in F_{k,t-r}^n$ such that $\tilde{a}$ is greater $\tilde{b}$ for some $\tilde{a} \in A$ and denote it by $T_r(A)$. Denote for brevity $T(A) = T_1(A)$.

The main problem we investigate here (denote it by Z) is to find for each n, k, t, r and $m = 1, ..., |F_{k,t}^n|$ a subset $A_0 \subseteq F_{k,t}^n$ such that $|A_0| = m$ and $|T(A_0)| \leq |T(A)|$ for any $A \subseteq F_{k,t}^n$, $|A| = m$.

The problems of such a type were investigated in a lot of papers. For the case $k = 0$, i.e. when F_k^n is isomorphic to the n-dimensional unit cube, the problem Z was solved in [2] for example. For $k = 1$, i.e. when F_k^n is isomorphic to the lattice of faces on the n-cube ordered by inclusion, the problem Z was solved in [4] (see also [3]). But in the last case the solution was obtained for $r \geq 1$ only. For $r < 0$ an additional arguments are required as in contradistinction to the case $k = 0$, the semilattice F_k^n is not rank symmetric.

In Section 2 of this paper we present the main result of it, that solves the problem Z in the case $r = 1$ (see Theorem 1). Section 3 is devoted to some corollaries of the main result. We show there that from Theorem 1 the solution of Z in the case $r > 1$ follows immediately (see Theorem 2) and investigate the case $r < 0$. In this case we obtain the general property of ranked posets (see Theorem 3) which allows us to deduce the solution of Z from the case $r \geq 1$ to the case $r < 0$. Besides of it we construct there ideals of F_k^n, that have minimal possible weight in the class of monotone weight functions (see Theorem 4). Such problem for $k = 0$ and $k = 1$ was solved in [1] and [4] respectively.

2 The Main Result

Now we are going to describe the ordering L_k^n of the set F_k^n. For this purpose we at first describe the ordering N_k^n of the set $F_{k,0}^n$, where

$$F_{k,0}^n = \{(a_1, ..., a_n) \in F_k^n : a_i \in \{0, ..., k\}, \quad 1 \le i \le n\}.$$

Denote by FF_k^n the set of vertices obtained from F_k^n by replacing the symbol 0 to 1, 1 to 2,..., $k-1$ to k and let

$$H_k^n = \{(a_1, ..., a_n) \in F_{k,0}^n : a_i \in \{0, k\}, \quad 1 \le i \le n\}.$$

The definition of N_k^n is inductive on k. Notice that $|F_{0,0}^n| = 1$, so the order N_0^n is already defined.

Inductive basis. Let N_1^n be the lexicographic order for any n.

Inductive step. Let us assume that for each k', $1 \le k' \le k$, the order $N_{k'}^n$ over the set $F_{k'}^n$ (and, therefore over $FF_{k'}^n$) is defined, and consider the case $k' = k$.

Let the vertex $\tilde{0} = (0, ..., 0)$ be the least in N_k^n. We say that it forms the first block $B_1 = \{\tilde{0}\}$. The definition of N_k^n is in fact the double induction on k and on the number of block. Assume that we have ordered all the vertices of $F_{k,0}^n$ included to the blocks $B_1, ..., B_j$ and $j < 2^n$. Denote by $\tilde{a} \in H_k^n$ the greatest (in N_k^n) vertex of the block B_j and let $\tilde{b} \in H_k^n$ be the next vertex in the lexicographic order. Consider the collection $B_{j+1} \subseteq F_{k,0}^n$ of vertices, obtained from $\tilde{b}$ by replacing of each $b_i \neq 0$ to an arbitrary symbol from the set $\{1, ..., k\}$ $\quad i = 1, ..., n$. Let $\tilde{c}_1, \tilde{c}_2 \in B_{j+1}$ and s be the number of zero coordinates of $\tilde{b}$. Denote by $\tilde{c'}_1, \tilde{c'}_2$ the vertices obtained from $\tilde{c}_1, \tilde{c}_2$ respectively by deleting the s zero entries. Notice that $\tilde{c'}_1, \tilde{c'}_2 \in FF_{k,0}^{n-s}$. Let $\tilde{c}_1 < \tilde{c}_2$ in N_k^n iff $\tilde{c'}_1 < \tilde{c'}_2$ in N_{k-1}^{n-s}. More than that let the least vertex of B_j be the next in N_k^n after $\tilde{a}$. We call $\tilde{b}$ the basic vertex of the block B_j.

As an example we present the ordering N_3^2 of $F_{3,0}^2$. In order to facilitate the denotations we omitted commas and parenthesis in vectors. The basic vectors of blocks are underlined. Since N_3^2 involves N_1^1, N_2^1, N_1^2, N_2^2, we present them too:

$$N_1^1 : \underbrace{\underline{0}}_{B_1} \ \underbrace{\underline{1}}_{B_2}; \qquad N_2^1 : \underbrace{\underline{0}}_{B_1} \ \underbrace{1 \ \underline{2}}_{B_2}; \qquad N_1^2 : \underbrace{\underline{00}}_{B_1} \ \underbrace{\underline{01}}_{B_2} \ \underbrace{\underline{10}}_{B_3} \ \underbrace{\underline{11}}_{B_4}$$

$$N_2^2 : \underbrace{\underline{00}}_{B_1} \ \underbrace{01 \ \underline{02}}_{B_2} \ \underbrace{10 \ \underline{20}}_{B_3} \ \underbrace{11 \ 12 \ 21 \ \underline{22}}_{B_4} \text{ and, finally,}$$

$$N_3^2 : \underbrace{\underline{00}}_{B_1} \ \underbrace{01 \ 02 \ \underline{03}}_{B_2} \ \underbrace{10 \ 20 \ \underline{30}}_{B_3} \ \underbrace{11 \ 12 \ 13 \ 21 \ 31 \ 22 \ 23 \ 32 \ \underline{33}}_{B_4}.$$

Now we are ready to define the order L_k^n. If $k = 0$, then we let L_0^n be equal to the lexicographic order (with $-1 > 0$). If $k > 0$, then for $\tilde{a} = (a_1, ..., a_n) \in F_k^n$ denote by $[\tilde{a}]$ ($]\tilde{a}[$) the vector obtained from $\tilde{a}$ by replacing of each $a_i = -1$ to 0 (to k). For $\tilde{a}, \tilde{b} \in F_k^n$ we say that $\tilde{a} < \tilde{b}$ in order L_k^n iff

(i) $]\tilde{a}[<]\tilde{b}[$ in order N_k^n or

(ii) if $]\tilde{a}[=]\tilde{b}[$ holds, then $[\tilde{a}] > [\tilde{b}]$ in order N_k^n.
Later if it does not tend to contradiction we will omit the names of orders writing simply $\tilde{a} < \tilde{b}$. Let $\tilde{a} \in F_k^n$ and $]\tilde{a}[\in B_i$. Denote by $\hat{a} \in F_{k,0}^n$ the basic vector of B_i.

Lemma 1 *The following properties of L_k^n hold:*

1. If $\tilde{a}, \tilde{b} \in F_k^n$ and $\hat{a} < \hat{b}$ then $\tilde{c} < \tilde{d}$ for any $\tilde{c}, \tilde{d}$ such that $\hat{c} = \hat{a}$, $\hat{d} = \hat{b}$;

2. If $\tilde{a}, \tilde{b} \in F_k^n$ and $a_i = b_i$ for some i then $\tilde{a} < \tilde{b}$ iff $\tilde{a'} < \tilde{b'}$, where $\tilde{a'}, \tilde{b'}$ are obtained from $\tilde{a}, \tilde{b}$ respectively by deleting the i-th entry.

3. If $\tilde{a} = (a_1, ..., a_n) \in F_k^n$ and $a_i \notin \{k, -1\}$ for some i, then $\tilde{a} < \tilde{b}$, where $\tilde{b}$ is obtained from $\tilde{a}$ by replacing a_i to $a_i + 1$.

4. If $\tilde{a} = (a_1, ..., a_n) \in F_k^n$ and $a_i = -1$, $a_j = k$ for some $i < j$, then $\tilde{a} > \tilde{b}$, where $\tilde{b}$ is obtained from $\tilde{a}$ by replacing a_i to k and a_j to -1.

Proof.
The validity of the Property 1 immediately follows from the definition of L_k^n.

Now we prove Property 2 using induction on k. For $k = 0$ and arbitrary n it is obviously true. Let us make the inductive step. Notice that $\hat{a} < \hat{b}$ iff $\hat{a'} < \hat{b'}$. Therefore if $]\tilde{a}[$ and $]\tilde{b}[$ belong to different blocks then $\tilde{a'} < \tilde{b'}$ by Property 1. Let $]a[$ and $]b[$ belong to the same block.

If $a_i = b_i = 0$ then $\tilde{a'} < \tilde{b'}$ by the definition of L_k^n. If $a_i = b_i \neq 0$ then consider $\tilde{a''}, \tilde{b''} \in F_{k'}^n$ obtained by deleting all zero entries of $\tilde{a}$ and $\tilde{b}$ respectively. Then $\tilde{a} < \tilde{b}$ iff $\tilde{a''} < \tilde{b''}$ by the definition of L_k^n. Using now the inductive hypothesis for $k' \leq k - 1$, we have that $\tilde{a''} < \tilde{b''}$ iff $\tilde{a'} < \tilde{b'}$.

The Property 3 may be proved analogously. In order to prove the Property 4, it is sufficient to notice that $]\tilde{a}[=]\tilde{b}[$ but $[\tilde{a}] < [\tilde{b}]$, since $]([\tilde{a}])[<]([\tilde{b}])[$. Hence $\tilde{a} > \tilde{b}$. ∎

Let $A \subseteq F_{k,t}^n$ and i, τ be integers, $1 \leq i \leq n$, $\tau \in \{0, ..., k, -1\}$. Denote

$$F_{k,t}^{n;\tau}(i) = \{(a_1, ..., a_n) \in F_{k,t}^n : a_i = \tau\},$$
$$A^\tau(i) = A \bigcap F_{k,t}^{n;\tau}(i).$$

Furthermore, denote by CA ($CA \subseteq F_{k,t}^n$) the set obtained by replacing A to the first $|A|$ vectors of $F_{k,t}^n$ in order L_k^n. Analogously let C_iA be the set obtained by replacing $A^\tau(i)$ to the first $|A^\tau(i)|$ vectors of $F_{k,t}^{n;\tau}(i)$ in order L_k^n, $\tau = -1, 0, ..., k$. We say that A is i-compressed if $C_iA = A$.

Theorem 1 *Let $A \subseteq F_{k,t}^n$. Then $T(CA) \subseteq CT(A)$.*

In order to prove this Theorem we need some lemmas.

Lemma 2 *Let $A \subseteq F_{k,t}^n$, $A \neq \emptyset$, $n \geq 3$ and $C_iA = A$ for $1 \leq i \leq n$. Then $\tilde{b} \in A$ where $\tilde{b}$ is the first vector of $F_{k,t}^n$ in order L_k^n.*

Proof.
For $t = 0$ and $t = n$ the lemma is obviously true. So let $1 \leq t < n$. Then $\tilde{b} = (0, ..., 0, \underbrace{-1, ..., -1}_{t})$. Let $\tilde{a} \in A$, $\tilde{a} \neq \tilde{b}$. Consider $\tilde{c} = [\tilde{a}]$. Then $\hat{c} \leq \hat{a}$, $\tilde{c} \leq \tilde{a}$ and $\tilde{c} \in A$,

since $t \geq 1$ and, therefore, the vectors $\tilde{a}$ and $\tilde{c}$ have at least one equal entry in common. If there exist i such that $c_i = b_i$, then $\tilde{b} \in A$ by Property 2 and equality $C_iA = A$. In opposite case $\tilde{c} = (\underbrace{-1, ..., -1}_{t}, 0, ..., 0)$. Consider vector $\tilde{d}$ obtained from $\tilde{b}$ by replacing b_1 to -1 and b_n to 0. Then $\tilde{b} < \tilde{d} < \tilde{c} \leq \tilde{a}$ and $\tilde{d} \neq \tilde{c}$, since $n \geq 3$. More than that, $d_1 = c_1$ and, therefore $\tilde{d} \in A$. Hence $\tilde{b} \in A$. ∎

Lemma 3 *Let $A \subseteq F^n_{k,t}$ and $T(CB) \subseteq CT(B)$ for any $B \subseteq F^{n-1}_{k,t'}$, $t' \leq t$. Then $T(C_i(A)) \subseteq C_iT(A)$ for $i = 1, ..., n$.*

Proof.
Let us fix the index i and consider the set $A^{-1}(i)$. Let $\tau \in \{0, ..., k\}$. Denote by $B^\tau \subseteq F^n_{k,t-1}$ the collection of all vectors obtained from the vectors $\tilde{a} \in A^{-1}(i)$ by replacing a_i to τ. It is easy to see that

$$|B^\tau| = |A^{-1}(i)|, \quad B^\tau \subseteq T(A) \bigcap F^{n,\tau}_{k,t-1}(i).$$

Furthermore, $C_i(A^{-1}(i))$ and $C_i(T(A) \cap F^{n,\tau}_{k,t-1}(i))$ are the initial segments in L^{n-1}_k of subsets $F^{n,-1}_k(i)$ and $F^{n,\tau}_k(i)$ respectively. Since $|T(A) \cap F^{n,\tau}_{k,t-1}(i)| \geq |A^{-1}(i)|$ and $T(CA^\tau(i)) \subseteq CT(A^\tau(i))$ for any $\tau \in \{0, ...k, -1\}$, then $T(C_iA) \subseteq C_i(T(A))$. ∎

Corollary 1 *After applying sufficiently many times operator C_i to $A \subseteq F^n_{k,t}$ for $i = 1, ..., n$, one obtains a subset $B \subseteq F^n_{k,t}$, which is i-compressed for any $i = 1, ..., n$ and $|T(B)| \leq |T(A)|$.* ∎

Lemma 4 *If $A \subseteq F^n_{k,t}$ and $CA = A$, then $CT(A) = T(A)$.*

Proof.
The Lemma is true for $n = 1$ and arbitrary k. Let $n \geq 2$. We prove the Lemma using induction on k. Notice that for $k = 0$ and arbitrary n the Lemma is true by [2]. Let $\tilde{e}, \tilde{f} \in F^n_{k,t-1}$, $\tilde{e} < \tilde{f}$, $\tilde{f} \in T(\tilde{g})$, $\tilde{g} \in A$. It is sufficient to prove that $\tilde{e} \in T(\tilde{c})$ for some $\tilde{c} \in F^n_{k,t}$, $\tilde{c} \leq \tilde{g}$.

Case 1. Assume $]\tilde{e}[<]\tilde{f}[$.

(*i*) Let $]\hat{e}[\neq]\tilde{f}[$. It means that $f_s \neq 0$, $e_s = 0$ for some s, $1 \leq s \leq n$, and $e_i = 0$ iff $f_i = 0$ for any $i < s$.

If $e_i \neq -1$ for some $s > i$, then consider vector $\tilde{c}$ obtained from $\tilde{e}$ by replacing e_i to -1. Then $\tilde{c} \in F^n_{k,t}$ and $\tilde{c} < \tilde{g}$ by Property 1, since $\hat{c} < \hat{g}$.

If $e_i = -1$ for all $i > s$ and $e_q \notin \{0, -1\}$ for some $q < s$, then replacing e_q to -1, we obtain vector $\tilde{c} \in F^n_{k,t}$, such that $\tilde{c} = \tilde{e}$ and $\tilde{c} < \tilde{g}$, since $\hat{c} < \hat{g}$. Let now $e_i = -1$ for all $i > s$ and $e_q \in \{0, -1\}$ for all $q < s$, i.e. $e_j \in \{0, -1\}$ for all j, $1 \leq j \leq n$.

Since $\tilde{e}, \tilde{f} \in F^n_{k,t-1}$, then if $f_s \neq -1$, we get $f_j = e_j$ for all $j \neq s$ and, therefore, replacing e_s to -1 we obtain vector $\tilde{c} \in F^n_{k,t}$, such that $\tilde{c} \leq \tilde{g}$. Otherwise, if $f_s = -1$, then $e_q = -1$, $f_q \notin \{0, -1\}$ for some $q \neq s$ and $f_j = e_j$ for any $j \notin \{s, q\}$. Considering again the vector $\tilde{c}$, obtained from $\tilde{e}$ by replacing e_s to -1, we get $\tilde{c} \leq \tilde{g}$.

(ii) Let $]\hat{e}[=]\hat{f}[$. It means that $e_i = 0$ iff $f_i = 0$. If $\tilde{g}$ was obtained from $\tilde{f}$ by replacing $f_i = 0$ to -1 for some i, then consider the vector $\tilde{c}$, obtained from $\tilde{e}$ by replacing $e_i = 0$ to -1. By Property 2, $\tilde{c} < \tilde{g}$.

Let $\tilde{g}$ was obtained from $\tilde{f}$ by replacing $f_i \notin \{0, -1\}$ to -1 for some i. Denote by s the number of common zeros in $\tilde{e}$ and $\tilde{f}$. Notice that $g_i = 0$ iff $e_i = 0$ (or $f_i = 0$). Consider the subset FF_k^{n-s}, and denote by $\tilde{e'}\tilde{f'}, \tilde{g'} \in FF_k^{n-s}$ the vectors, obtained from $\tilde{e}, \tilde{f}, \tilde{g}$ respectively by deleting the s common zeros. Then $\tilde{e'} < \tilde{f'}$ by Property 2. Using the inductive hypothesis, one gets that there exists a vector $\tilde{c'} \in FF_{k,t}^{n-s}$, such that $\tilde{c'} < \tilde{g'}$ and $\tilde{e'} \in T(\tilde{c'})$. Now put the zeros in $\tilde{c'}$ into their proper places. We obtain vector $\tilde{c} \in F_{k,t}^n$ and $\tilde{c} < \tilde{g}$ by Property 2.

Case 2. Assume $]\tilde{e}[=]\tilde{f}[$. Then $[\tilde{e}] > [\tilde{f}]$. Notice that $\hat{e} \neq \hat{f}$. Denote by j_1 and j_2 respectively the maximal and minimal index j such that $e_j = k$ and $f_j = -1$ and let $\tilde{g}$ obtained from $\tilde{f}$ by replacing f_s to -1. Then $j_1 < s$.

If $s_s = -1$ then $f_s = k$. Consider vector $\tilde{c}$ obtained from $\tilde{e}$ by replacing e_{j_2} to -1. If $j_1 = j_2$, then $\tilde{c} = \tilde{g}$. If $j_1 < j_2$ then $]\tilde{c}[=]\tilde{g}[$, but $[\tilde{c}] > [\tilde{g}]$, i.e. $\tilde{c} > \tilde{g}$.

If $e_s \neq -1$, then since $f_s \neq -1$ and $]\tilde{e}[=]\tilde{f}[$, then $e_s = f_s$. Consider vector $\tilde{c}$, obtained from $\tilde{e}$ by replacing e_s to -1. Using Property 2, it is easy to verify that $\tilde{c} < \tilde{g}$. ∎

Lemma 5 *Let $A \subseteq F_{k,t}^2$. Then $T(CA) \subseteq CT(A)$.*

Proof.
We use induction on k. For $k = 0, 1$ the Lemma is obviously true. Let $A \subseteq F_{k,t}^2$. Using Lemma 3, we may assume without loss of generality that A is i-compressed for $i = 1, 2$.

If $t = 0$ or $t = 2$ then the Lemma is obviously true. So let $t = 1$ and denote by $\tilde{a} \in F_{k,1}^2$ the last vector of A in order L_k^2. Since A is i-compressed then either $(-1, 0) \in A$ or $(0, -1) \in A$. Assume for example that $(0, -1) \in A$. Then if $(-1, 0) \notin A$ then any vector $\tilde{c} \in A$ is of the form $(c_1, -1)$ where $c_1 \in \{1, 2, ..., k\}$. However, in this case $T(\tilde{c'}) \cap T(\tilde{c''}) = \emptyset$ for any $\tilde{c'}, \tilde{c''} \in A$. But $T(0, -1) \cap T(-1, 0) \neq \emptyset$. Therefore, after replacing $\tilde{a}$ to $(-1, 0)$ we obtain an i-compressed subset A', $i = 1, 2$, such that $\{(-1, 0), (0, -1)\} \in A'$ and $|T(A')| \leq |T(A)|$.

Notice that

$$\begin{aligned} T' &= \{(c_1, c_2) \in F_{k,1}^2 : c_1 \cdot c_2 = 0\} \subseteq T(A') \quad \text{and} \\ A'' &= A' \backslash \{(-1, 0), (0, -1)\} \subseteq FF_k^2. \end{aligned}$$

Applying the inductive hypothesis to the set A'' and the sublattice FF_k^2, which is isomorphic to F_{k-1}^2, we get $T(CA'') \subseteq CT(A'')$ (in FF_k^2). Consider now the sets CA'' and $T(A'')$ in $F_{k,1}^2$. Then the desired inclusion $T(CA) \subseteq CT(A)$ easily follows from the equalities

$$\{(-1, 0), (0, -1)\} \cup CA'' = CA,$$

$$T' \cup CT(A'') = CT(A),$$

$$|T(A)| > |T(A')| = |T' \cup CT(A'')|.$$

∎

Proof of Theorem 1. We use double induction on n and k. For $n = 1$ and arbitrary k the Theorem is obviously true. For $n = 2$ and arbitrary k the Theorem is also true by Lemma 5. For $k = 0$ and arbitrary n the Theorem is equivalent to the Kruskal-Katona theorem which is true either [2]. Assume that the Theorem is true for any $k' < k$, $n' \leq n$ and for $k' = k$, $n' < n$ and consider the case $k' = k$ and $n' = n$. Without loss of generality we may assume that $k \geq 1$, $n \geq 3$.

Let $A \subseteq F^n_{k,t}$ and $t \geq 1$. Due to the corollary of Lemma 3 we may assume that A is i-compressed for $i = 1, ..., n$. Denote by $\tilde{b} \in F^n_{k,t}$ the first vector of $F^n_{k,t} \backslash A$ in order L^n_k and by $\tilde{a}$ the last vector of A in L^n_k. If $\tilde{b} > \tilde{a}$ then $CA = A$ and the Theorem is true. So let $\tilde{b} < \tilde{a}$. If $a_i = b_i$ for some i then $\tilde{b} \in A$ by Property 2, since A is i-compressed. Therefore let $a_i \neq b_i$, $i = 1, ..., n$.

In order to proof the Theorem it is sufficient to prove that after replacing $\tilde{a}$ to $\tilde{b}$ we obtain a subset $A' \subseteq F^n_{k,t}$, such that $|T(A')| \leq |T(A)|$. Indeed, if it is so then after the finite number of replacements we may convert A into CA without increasing $|T(A)|$, from where by Lemma 4 we get $T(CA) \subseteq CT(A)$.

Case 1. Let $\hat{a} < \hat{b}$. It means that there exists an index s, $1 \leq s \leq n$, for which $\hat{a}_1 = \hat{b}_1 = k$, $\hat{a}_2 = \hat{b}_2 = k$,..., $\hat{a}_{s-1} = \hat{b}_{s-1} = k$, $\hat{a}_s = k$, $\hat{b}_s = 0$.

Remark 1 *Without loss of generality we may assume that* $]a_i[\leq]b_i[$ *for* $i \neq s$, *since otherwise* $\tilde{b} \in A$.

Indeed, let $]a_i[>]b_i[$. Then $b_i \notin \{k, -1\}$. If $a_i \neq -1$, then consider vector $\tilde{c}$ obtained from $\tilde{b}$ by replacing b_i to $]a_i[$. By Property 3, $\tilde{b} < \tilde{c}$. More than that, $\hat{c} < \hat{a}$, i.e. $\tilde{c} < \tilde{a}$ by Property 1 and $a_i = c_i$, $b_s = c_s$. Therefore, taking into consideration Property 2 and that A is i, s-compressed, we get $\tilde{c} \in A$ and $\tilde{b} \in A$. Now if $a_i = -1$, then let $b_j = -1$, $j \notin \{i, s\}$. Consider vector $\tilde{c}$ obtained from $\tilde{b}$ by replacing b_j to k and b_i to -1. Since $]\tilde{b}[<]\tilde{c}[$ then $\tilde{b} < \tilde{c}$. Moreover, $c < a$ and $c_s = b_s$ and $a_i = c_i$. Hence $\tilde{c} \in A$ and $\tilde{b} \in A$.

Remark 2 *If* $b_n = k$ *then* $T(\tilde{b}) \subseteq T(A \backslash \tilde{b})$.

Indeed, any vector $\tilde{b'} \in T(\tilde{b})$ may be obtained from $\tilde{b}$ by replacing some $b_i = -1$ to $c \in \{0, 1, ..., k\}$. Consider vector $\tilde{c}$ obtained from $\tilde{b}$ by replacing $b_i = -1$ to c and b_n to -1. By Properties 3,4 $\tilde{c} < \tilde{b}$, hence $\tilde{c} \in A$. But $\tilde{b'} \in T(\tilde{c})$, therefore after replacing $\tilde{a}$ to $\tilde{b}$, $|T(A)|$ cannot increase.

Remark 3 *Without loss of generality we may assume that there do not exist indexes* i, j *such that* $i < s < j$ *and* $a_i = b_j = 1$, *since then* $\tilde{b} \in A$.

Indeed, if such indexes exist and $b_i \neq k$ then $\tilde{b} \in A$ by Remark 1. If $b_i = k$, then consider vector $\tilde{c}$, obtained from $\tilde{b}$ by replacing b_i to -1 and b_j to k. We get $\tilde{c} > \tilde{b}$ by Property 3 and $\tilde{c} \in A$ since $\tilde{c} < \tilde{a}$, $n \geq 3$. Hence $\tilde{b} \in A$.

Taking into account remarks 1–3 consider the following two subcases:

(i) Let there exists index j, $s < j \leq n$, such that $a_j \neq 0$. If $a_n = 0$, then consider vector $\tilde{c}$ obtained from $\tilde{a}$ by replacing a_j to 0 and a_n to a_j. If $a_n \neq 0$ then let $\tilde{c} = \tilde{a}$. In any case $\tilde{b} < \tilde{c} \leq \tilde{a}$ and $\tilde{c} \in A$, since $c_s = a_s$.

Now we are going to show that there exists vector $\tilde{d} \in F_{k,t}^n$, such that $\hat{d} = \hat{c}$, $\tilde{d} \leq \tilde{c}$ and $d_n = -1$. Indeed, if $c_n) = -1$, then let $\tilde{d} = \tilde{c}$. If $c_n \neq -1$ and $c_i = 0$ exactly for $q \geq 1$ indexes i, $1 \leq i \leq n$, then consider vector $\tilde{c'} \in FF_{k,t}^{n-q}$ obtained from $\tilde{c}$ by deleting all zeros. Denote by $\tilde{d'}$ the first vector of $FF_{k,t}^{n-q}$ in L_k^{n-q} and consider vector $\tilde{d} \in F_{k,t}^n$ obtained by inserting zeros into $\tilde{d'}$ on their proper places. Then $\tilde{d} < \tilde{c}$, since $\tilde{d'} < \tilde{c'}$ by Property 2. Since $q \geq 1$, then $\tilde{d} \in A$ and obviously $\hat{d} = \hat{c}$. Finally, if $c_n \neq -1$ and $c_i \neq 0$ for $1 \leq i \leq n$, then consider subset $A_0 = \{(a_1, ...a_n) \in A : a_i \neq 0,\ 1 \leq i \leq n\} \subseteq FF_{k,t}^n$. It is not difficult to show that A_0 is i-compressed for all i and since $n \geq 3$ then $d = (1, ..., 1, \underbrace{-1, ..., -1}_{t}) \in A_0$ by Lemma 3. We also have $\tilde{d} \in A$, $\tilde{d} \leq \tilde{c}$, $\hat{d} = \hat{c}$, therefore $\tilde{b} < \tilde{d}$ by Property 1.

Now if $b_n = -1$ then $\tilde{b} \in A$ since $\tilde{d} \in A$. If $b_n \notin \{k, -1\}$ then $\tilde{b} \in A$ by Remark 1. Finally, if $b_n = k$ then $|T((A\backslash\tilde{a}) \cup \tilde{b})| \leq |T(A)|$ by Remark 2.

(ii) Let $a_j = 0$ for $s < j \leq n$ or $s = n$. Consider the case when there exists vector $\tilde{c} \in A$ such that $c_{s-1} = -1$, $\hat{c} = \hat{a}$, $\tilde{c} < \tilde{a}$. If now $b_j = -1$ for some j, $s < j \leq n$, then $\tilde{b} \in A$ by Remark 3. If $b_{s-1} \neq k$ then $\tilde{b} \in A$ by Remark 1. Finally, if $b_{s-1} = k$ then taking into account $b_j \neq -1$ for $j > s$, it is easy to show that $|T(A\backslash\tilde{b})| = |T(A)|$ (see the arguments from Remark 2).

Let now such a vector $\tilde{c}$ does not exist. We are going to show that it is possible only if $t = 1$ and $\tilde{a} = (\underbrace{1, ..., 1, -1}_{s}, 0, ...0)$. Indeed, in the case $t \geq 2$ if $s < n$ then $\tilde{c} = (\underbrace{1, ..., 1, \overbrace{-1, ..., -1}^{t}}_{s}, 0, ..., 0) \in A$ since $c_n = a_n = 0$ and $\tilde{c} < \tilde{a}$. If $s = n$ then, applying Lemma 3 to the set $FF_{k,t}^n$, we conclude that $\tilde{c} = (1, ...1, \overbrace{-1, ..., -1}^{t}) \in A$. In any case $\tilde{c} = \tilde{a}$ and $c_{s-1} = -1$.

On the other hand if $t = 1$ and $\tilde{a} \neq (\underbrace{1, ..., 1, -1}_{s}, 0, ...0)$, then similarly to above $\tilde{c} = (\underbrace{1, ..., 1, -1}_{s}, 0, ...0) \in A$. Notice that the next vector after the vector $\tilde{c}$ in order L_k^n is the vector $\tilde{d} = (\underbrace{1, ..., 1, -1, 1}_{s}, 0, ..., 0)$. The condition $a_s \neq -1$ in this case means that $\tilde{c} < \tilde{d} \leq \tilde{a}$ and since $n \geq 3$ then $\tilde{d} \in A$. However $d_{s-1} = -1$.

Therefore the only case should be considered now is $\tilde{a} = (\underbrace{1, ..., 1, -1}_{s}, 0, ..., 0)$. We are going to show that $|T(A)| - |T(A\backslash\tilde{a})| \geq k$. Indeed, $|T(\tilde{d})| = k + 1$ for any $\tilde{d} \in F_{k,1}^n$. Denote by $\tilde{a'} \in F_{k,0}^n$ arbitrary vector obtained from $\tilde{a}$ by replacing a_s to $c \in \{1, ..., k\}$. Then $\tilde{a'} \in T(\tilde{a})$ and the conditions $\tilde{a'} \in T(\tilde{c})$, $\tilde{c} \in F_{k,1}^n$ imply $\tilde{c} > \tilde{a}$, i.e. $\tilde{c} \notin A$. We show now that $|T(A)| - |T(A\backslash\tilde{b})| \leq k$. Let $b_i = -1$ and $\tilde{b'} \in T(\tilde{b})$ be the vector obtained from $\tilde{b}$ by replacing b_i to 0. If $b_j \neq 0$ for some $j \neq i$, then obtain vector $\tilde{c}$ by replacing b_j to -1. Then $\tilde{b'} \in T(\tilde{c})$, $\tilde{c} < \tilde{b}$ and hence, $\tilde{c} \in A$. Contrary if $b_j = 0$ for all $j \neq i$, then denote $\tilde{c} = (0, ..., 0, -1)$. Then $\tilde{b'} \in T(\tilde{c})$, $\tilde{c} \leq \tilde{b}$ and either $\tilde{b} \in A$ by Lemma 3, when $\tilde{c} = \tilde{b}$, or $\tilde{c} \in A$ when $\tilde{c} < \tilde{b}$. Therefore, $|T((A\backslash\tilde{a}) \cup \tilde{b}| \leq |T(A)|$.

Case 2. Let $\hat{a} = \hat{b}$. If $\tilde{b} \neq (k, ..., k)$, then $a_i = b_i = 0$ for some i, i.e. $\tilde{b} \in A$. Contrary, if $\tilde{b} = (k, ..., k)$, then it means that $A_0 = \{(a_1, ..., a_n) \in F^n_{k,t} : a_i = 0 \text{ for some } i\} \subseteq A$ and, respectively, $T_0 = \{(a_1, ..., a_n) \in F^n_{k,t-1} : a_i = 0 \text{ for some } i\} \subseteq T(A)$. Therefore, $A_1 = A \backslash A_0 \subseteq FF^n_{k,t}$. Using now the arguments of the proof of Lemma 5 and the inductive hypothesis for the set $FF^n_{k,t}$, we get $T(CA_1) \subseteq CT(A_1)$. Now consider the subsets CA_1 and $T(A_1)$ in $F^n_{k,t}$. Taking into account the equalities $A_0 \cup CA_1 = CA$ and $T_0 \cup CT(A_1) = CT(A)$ and using the definition of the order L^n_k, we get $T(CA) \subseteq CT(A)$. ∎

3 Some Consequences of the Main Theorem

1. Let $r \leq t - 1$.

Theorem 2 $T_r(CA) \subseteq CT_r(A)$ *for any* $r \geq 1$.

In order to proof the Theorem it is sufficient to apply Theorem 1 to the set A, then to the set $T(A)$, $T_2(A)$ and so forth, until the set $T_r(A)$. ∎

2. Denote by $P_r(A)$ the collection of all $\tilde{b} \in F^n_{k,t+r}$, such that $\tilde{b}$ is greater $\tilde{a}$ in L^n_k for some $\tilde{a} \in A$. Denote for brevity $P(A) = P_1(A)$. Consider the problem to find for each n, k, t, r and $m = 1, ..., |F^n_{k,t}|$ a subset $A_0 \subseteq F^n_{k,t}$, such that $|A_0| = m$ and $|P(A_0)| \leq |P(A)|$ for any $A \subseteq F^n_{k,t}$, $|A| = m$. For $A \subseteq F^n_{k,t}$ denote by $LA \subseteq F^n_{k,t}$ the set obtained from A by replacing it to the last $|A|$ elements of $F^n_{k,t}$, ordered by L^n_k.

Theorem 3
(*i*) $P(LA) \subseteq LP(A)$;
(*ii*) *if* $LA = A$, *then* $LP(A) = P(A)$.

Proof.
(*i*) Let us assume that $F^n_{k,t+1} \neq \emptyset$. Denote $\bar{A} = F^n_{k,t} \backslash A$ and $B = \{\tilde{b} \in F^n_{k,t+1} : T(\tilde{b}) \subseteq A\}$. Notice that $T(CB) \subseteq C\bar{A}$. If we replace $P\bar{A}$ to $C\bar{A}$ and B to CB then A will be replaced to LA and $P(A)$ to $LP(A)$. Since $LA = F^n_{k,t} \backslash C\bar{A} \subseteq F^n_{k,t} \backslash T(CB)$ and $T(CB) \cap (F^n_{k,t} \backslash T(CB)) = \emptyset$, then $LA \cap T(CB) = \emptyset$ and $PL(A) \subseteq LP(A)$.

(*ii*) Using the previous notations it is sufficient to prove that $CB = B$. Denote by $\tilde{b}$ the first vector of $F^n_{k,t} \backslash B$ in L^n_k and by $\tilde{a}$ the last vector of B. If $\tilde{a} < \tilde{b}$ then (*ii*) is true. Assume $\tilde{a} > \tilde{b}$. Then $\tilde{b} \notin B$ implies $T(\tilde{b}) \cap A = \emptyset$. Since $T(B) \subseteq \bar{A}$ and $C\bar{A} = \bar{A}$, then $T(CB) \subseteq \bar{A}$ by Theorem 1. However, $\tilde{b} \in CB$ therefore $T(\tilde{b}) \subseteq A$ i.e. $T(\tilde{b}) \cap A \neq \emptyset$. A contradiction. Hence, $\tilde{b} > \tilde{a}$ i.e. $CB = B$. ∎

It is easy to see that for $r > 1$ the solution of our problem can be obtained by applying the arguments of the paragraph above.

3. Let $A \subseteq F^n_k$. Denote $A_t = A \cap F^n_{k,t}$ for $t = 0, ..., n$. The set A is called ideal if $T(A_t) \subseteq A_{t-1}$, $t = 1, ..., n$. Let $w_0 \leq w_1 \leq ... \leq w_n$ be arbitrary numbers and $w_0 \geq 0$. We call the magnitude

$$f(A) = \sum_{i=0}^{n} w_i \cdot |A_i|$$

the weight of A. Consider the problem to find for each n, k and $m = 1, ..., |F^n_k|$ an ideal $A_0 \subseteq F^n_k$, such that $|A_0| = m$ and $f(A_0) \geq f(A)$ for any ideal $A \subseteq F^n_k$, $|A| = m$. For

$A \subseteq F_k^n$ denote by $FA \subseteq F_k^n$ the set obtained from A by replacing it to the first $|A|$ elements of F_k^n, ordered by L_k^n. Later we will see that FA is ideal for any A.

Theorem 4 *If A is ideal then $f(A) \leq f(FA)$.*

In order to prove this Theorem we need the following

Lemma 6 *Let $A \subseteq F_k^n$ is ideal, $\tilde{a} \in A$ and $\tilde{b} \in F_k^n$ is such, that $\|\tilde{a}\| > \|\tilde{b}\|$ and $\tilde{a} > \tilde{b}$ in order L_k^n. Then $\tilde{b} \in A$.*

Proof.
We use induction on k. For $k = 0$ and arbitrary n the Lemma is true by [1]. Assume that it is true for all $k' < k$ and consider the case $k' = k \geq 1$.

Case 1. Let $]\tilde{a}[>]\tilde{b}[$. If $\hat{b} < \hat{a}$ then there exists an index i such that $a_i > b_i = 0$ and for for $j < i$ $b_j = 0$ iff $a_j = 0$. Consider vector $\tilde{c}$ obtained from $\tilde{a}$ by replacing arbitrary $\|\tilde{a}\| - \|\tilde{b}\|$ entries, that are equal -1, to k. Then $\|\tilde{b}\| = \|\tilde{c}\|$ and $\tilde{b} < \tilde{c}$ since $\hat{b} < \hat{c} = \hat{a}$. Hence $\tilde{b} \in A$.

If $\hat{b} = \hat{a}$ then obtain vectors $\tilde{a'}$ and $\tilde{b'}$ by deleting all zero coordinates from $\tilde{a}$ and $\tilde{b}$ respectively. Then $\tilde{a'}, \tilde{b'} \in F_{k-1}^{n'}$ for some $n' \leq n$, $\tilde{a'} > \tilde{b'}$ by Lemma 1 and $\|\tilde{a'}\| > \|\tilde{b'}\|$. Similarly we may transform each vector $\tilde{c} \in A$ for which $\tilde{c} = \tilde{a} = \tilde{b}$ and obtain in such a way a subset A'. Using the inductive hypothesis, we conclude that there exists $\tilde{c'} \in A$' such that $\tilde{b'} < \tilde{c'}$, $\|\tilde{b'}\| = \|\tilde{c'}\|$, hence $\tilde{b'} \in A'$. Then after inserting zeros into $\tilde{b'}$ and $\tilde{c'}$ to their proper positions, we get $\tilde{b} < \tilde{c}$ by Lemma 1 and $\|\tilde{b}\| = \|\tilde{c}\|$, i.e. $\tilde{b} \in A$.

Case 2. Let $]\tilde{a}[=]\tilde{b}[$. Then $[\tilde{b}] > [\tilde{a}]$ and $\hat{a} \neq \hat{b}$. It means that there exists an index i for which $a_i = -1$, $b_i = k$ and $a_j = b_j$ for all $j < i$. Denote by p the number of indices j, $j < i$, such that $a_j = b_j = -1$. Then there exist exactly $\|\tilde{a}\| - p - 1$ indices j, $j > i$, such that $a_j = -1$ and exactly $\|\tilde{b}\| - p$ indices q, $q > i$, such that $b_q = -1$. If $\|\tilde{b}\| - p \geq 1$, then $0 < \|\tilde{a}\| - \|\tilde{b}\| \leq \|\tilde{a}\| - p - 1$. Obtain vector $\tilde{c}$ from $\tilde{a}$ by replacing of arbitrary $\|a\| - \|b\|$ entries $a_j = -1$ to k, $j < i$. Then $]\tilde{c}[=]\tilde{b}[$, $\|\tilde{c}\| = \|\tilde{b}\|$ and $\tilde{c} > \tilde{b}$, since $c_i = -1$, $b_i = k$ and $c_j = b_j$, $j < i$. Since $\tilde{c} \in A$ then $\tilde{b} \in A$. Finally if $\|b\| = p$, then $a_j = b_j$ for all $j \neq i$ and therefore, $\tilde{b} \in T(\tilde{a})$, i.e. $\tilde{b} \in A$ since A is an ideal. ∎

Proof of Theorem 4. Let $A \subseteq F_k^n$. Consider the set $B = \cup_{i=1}^n CA_i$. From Theorem 1 it follows that B is an ideal. Denote by $\tilde{a}$ the last vector of B in L_k^n and by $\tilde{b}$ the first vector of $F_k^n \backslash B$ in L_k^n. If $\|\tilde{b}\| = \|\tilde{a}\|$, then $\tilde{b} \in B$ since $CB_i = B_i$ for all i. If $\|\tilde{b}\| > \|\tilde{a}\|$, then $\tilde{b} \in B$ by Lemma 6. Assume $\|\tilde{b}\| < \|\tilde{a}\|$, and denote $C = (B \backslash \tilde{a}) \cup \tilde{b}$. Since $T_s(\tilde{b}) \subseteq B$ for any $s \geq 1$ by the definition of $\tilde{b}$, then C is ideal too. However $f(C) > f(A) = f(B)$. If $C \neq FA$ then we may repeat the analogous replacement. Finally we will get $f(A) \leq f(FA)$. ∎

Theorems 2 and 3 imply that for $w_0 \geq w_1 \geq ... \geq w_n \geq 0$ the analog of Theorem 4 for filters in the poset F_k^n holds.

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