A Kruskal-Katona Type Theorem for the Linear Lattice

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Abstract

We present an analog of the well-known Kruskal-Katona theorem for the poset of subspaces of $\mathrm{PG}(n,2)$ ordered by inclusion. For given k,ℓ ($k<\ell$) and m the problem is to find a family of size m in the set of ℓ-subspaces of $\mathrm{PG}(n,2)$, containing the minimal number of k-subspaces. We introduce two lexicographic type orders $\mathcal{O}^1$ and $\mathcal{O}^2$ on the set of ℓ-subspaces, and prove that the first m of them, taken in the order $\mathcal{O}^1$, provide a solution in the case $k=0$ and arbitrary $\ell>0$, and one taken in the order $\mathcal{O}^2$, provide a solution in the case $\ell=n-1$ and arbitrary $k<n-1$. Concerning other values of k and ℓ, we show that for $n\geq 3$ the considered poset is not Macaulay by constructing a counterexample in the case $\ell=2$ and $k=1$.

1 Introduction

Denote by $\mathcal{L}$ the collection of all proper nonempty subspaces of $\mathrm{PG}(n,2)$ ordered by inclusion (cf. Fig. 1) and let $\mathcal{L}_\ell$ be the set of all ℓ-dimensional subspaces in $\mathcal{L}$, $\ell=0,\ldots,n-1$. For $B\subseteq\mathcal{L}_\ell$, $\ell>0$, and $k<\ell$ introduce the shadow of B:

$$\Delta_k^\ell(B)=\{a\in\mathcal{L}_k \mid a\subseteq b \text{ for some } b\in B\}.$$

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For fixed integers k, ℓ and m ($0 \leq k < \ell < n$, $1 \leq m \leq |\mathcal{L}_\ell|$) we consider the problem of finding an m-element set $B \subseteq \mathcal{L}_\ell$ minimizing $|\Delta_k^\ell(B)|$ among all m-element subsets of $\mathcal{L}_\ell$. We call such a set Δ_k^ℓ-optimal. We are particularly interested in the case when there exist nested Δ_k^ℓ-optimal sets $\{A_m\}$, i.e. such that $|A_m| = m$ and $A_{m-1} \subseteq A_m$, $m = 1, \ldots, |\mathcal{L}_\ell|$.

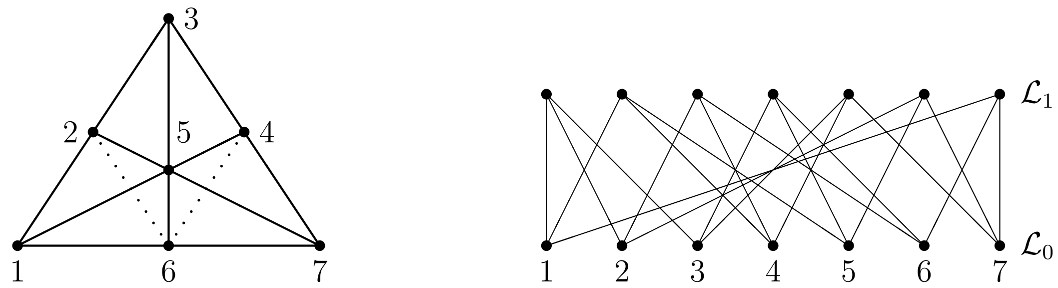

Figure 1: PG $(2, 2)$ and the corresponding poset $\mathcal{L}$.

Our problem may be considered as a natural extension of a similar problem for the Boolean poset B^n, the poset of subsets of an n-element set N ordered by inclusion. It is well known that our poset and the Boolean poset have some similar features. As it turns out, in some cases the solutions of the shadow minimization problem for both posets are, in a sense, also similar but in some other cases there is an essential difference. To make this more precise, we first recollect what is known about the shadow minimization problem for the Boolean poset. For more information on the subject the reader is referred to chapter 8 in the book [1].

Let $N = \{1, 2, \ldots, n\}$ and represent a subsets $A \subseteq N$ by binary characteristic vector $(\alpha_1, \ldots, \alpha_n)$. So $\alpha_i = 1$ if $i \in A$ and 0 otherwise. To a binary vector $\beta = (\beta_1, \ldots, \beta_t)$ we associate its lexicographic number

$$\text{lex}(\beta) = \sum_{i=1}^{t} \beta_i \cdot 2^{t-i},$$

and we say that α is greater than β in the lexicographic order if $\text{lex}(\alpha) > \text{lex}(\beta)$.

Theorem 1 (Kruskal [3], Katona [2]). *The family of ℓ-subsets corresponding to the first m characteristic vectors in the lexicographic order, together contain the minimal possible number of k-subsets, for any $k < \ell$.*

Hence, the lexicographic order provides nested solutions in the Boolean lattice.

We present an analog of this theorem for the poset $\mathcal{L}$, introducing linear orders $\mathcal{O}^1$ and $\mathcal{O}^2$ on $\mathcal{L}$, which are similar to the lexicographic order. The $2^{n+1}-1$ points of PG $(n, 2)$ are just the $(n+1)$-dimensional non-zero binary vectors $(\beta_1, \ldots, \beta_{n+1})$, and as before we associate with each point a lexicographic number. Using this ordering of the points, each subspace $a \in \mathcal{L}$ may be represented by its characteristic vector, i.e. by the $(2^{n+1} - 1)$-dimensional binary vector $(\alpha_{2^{n+1}-1}, \ldots, \alpha_1)$, where α_i corresponds to the i^{th} point of PG $(n, 2)$.

For two subspaces $a, b \in \mathcal{L}$, we say that a is greater than b in the order $\mathcal{O}^1$ (notation $a \succ_{\mathcal{O}^1} b$) iff the characteristic vector of a is greater than the one of b in the lexicographic order. The restriction of $\mathcal{O}^1$ to $\mathcal{L}_\ell$ is denoted by $\mathcal{O}^1_\ell$. Such an ordering is shown in Fig. 1 for PG $(2, 2)$, where the points are represented by their lexicographic numbers.

The order $\mathcal{O}^2$ is a bit more complicated. We define just its restriction $\mathcal{O}^2_\ell$ to $\mathcal{L}_\ell$ and let the order $\mathcal{O}^2$ be arbitrary linear extension of these orders. It is known that if one considers the set of hyperplanes of PG $(n, 2)$ as a collection of points, then it is possible to construct a new geometry PG $'(n, 2)$ on this set. We denote by $\mathcal{L}'$ the corresponding poset of subspaces of PG $'(n, 2)$. The two geometries PG $(n, 2)$ and PG $'(n, 2)$ are equivalent in the sense that there exists a bijection $\phi : \mathcal{L} \mapsto \mathcal{L}'$ such that $\phi(\mathcal{L}_\ell) = \mathcal{L}'_{n-1-\ell}$ for $\ell = 0, \dots, n-1$ and for any $a, b \in \mathcal{L}$ the inclusion $a \subseteq b$ holds iff $\phi(b) \subseteq \phi(a)$ in $\mathcal{L}'$. In other words, the Hasse diagrams of $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic. We put for each $\ell = 0, \dots, n-1$ and $i = 1, \dots, |\mathcal{L}_\ell|$ the i^{th} element of $\mathcal{L}_\ell$ in order $\mathcal{O}^2_\ell$ to be the element $\phi^{-1}(a)$, where $a \in \mathcal{L}'_{n-1-\ell}$ is the element with number $|\mathcal{L}'_{n-1-\ell}| - i + 1$ in the order $\mathcal{O}^1_{n-1-\ell}$ in $\mathcal{L}'$.

It should be mentioned that the orders $\mathcal{O}^1_\ell$ and $\mathcal{O}^2_\ell$ are different in general, while similar construction in the Boolean poset leads to two isomorphic (namely lexicographic) orders.

Denote by $\mathcal{O}^1_\ell(m)$ and $\mathcal{O}^2_\ell(m)$ the initial segments of $\mathcal{L}_\ell$ of length m, i.e the collection of the first m elements of $\mathcal{L}_\ell$ in the orders $\mathcal{O}^1_\ell$ and $\mathcal{O}^2_\ell$ respectively. Our main result is the following theorem:

Theorem 2

(i) $|\Delta^\ell_0(\mathcal{O}^1_\ell(m))| \leq |\Delta^\ell_0(B)|$ *for any* $B \subseteq \mathcal{L}_\ell$, $|B| = m$ *and* $0 < \ell \leq n-1$*;*

(ii) $|\Delta^{n-1}_k(\mathcal{O}^2_{n-1}(m))| \leq |\Delta^{n-1}_k(B)|$ *for any* $B \subseteq \mathcal{L}_{n-1}$, $|B| = m$ *and* $0 \leq k < n-1$.

The paper is organized as follows. First, for the sake of convenience, we dualize the problem. Then, in section 3 we solve the dual problem. In the last section we introduce Macaulay posets and show that the restrictions for k and ℓ in Theorem 2 are essential for the existence of nested Δ^ℓ_k-optimal subsets satisfying the Macaulay conditions. Moreover, we show that neither $\mathcal{O}^1$ nor $\mathcal{O}^2$ provide nested solutions for the whole poset $\mathcal{L}$. Here we have a difference with the Boolean poset, which is Macaulay and where just the lexicographic order works for all k and ℓ.

2 The dual problem

For $A \subseteq \mathcal{L}_k$ and $\ell > k$ we define

$$\Lambda^\ell_k(A) = \{b \in \mathcal{L}_\ell \mid \Delta^\ell_k(\{b\}) \subseteq A\}.$$

For given k and ℓ, with $\ell > k$, we consider the problem of finding a set $A \subseteq \mathcal{L}_k$ maximizing $|\Lambda^\ell_k(A)|$ among all subsets of $\mathcal{L}_k$ of the same cardinality. We call such a set Λ^ℓ_k-optimal.

Denote

$$\Delta_k^\ell(m) := |\Delta_k^\ell(B)|, \qquad \Lambda_k^\ell(m) := |\Lambda_k^\ell(A)|$$

for a Δ_k^ℓ-optimal set $B \subseteq \mathcal{L}_\ell$ and a Λ_k^ℓ-optimal set $A \subseteq \mathcal{L}_k$ with $|A| = |B| = m$. Similarly as above we define nested Λ_k^ℓ-optimal sets.

Lemma 1 *If there exist nested Λ_k^ℓ-optimal sets $A_m \subseteq \mathcal{L}_k$, $m = 0, \ldots, |\mathcal{L}_k|$, then there exist nested Δ_k^ℓ-optimal sets $B_p \subseteq \mathcal{L}_\ell$, $p = 0, \ldots, |\mathcal{L}_\ell|$.*

Proof.
We construct the subsets B_p using the subsets A_m. Assume

$$\Lambda_k^\ell(m-1) < \Lambda_k^\ell(m). \tag{1}$$

Let $\{b_1, \ldots, b_s\} = \Lambda_k^\ell(A_m) \setminus \Lambda_k^\ell(A_{m-1})$ for some $s > 0$. Then for any r, $0 < r \leq s$, the subset $\Lambda_k^\ell(A_{m-1}) \cup \{b_1, \ldots, b_r\}$ is Δ_k^ℓ-optimal. Indeed, if some of these subsets, say B, is not Δ_k^ℓ-optimal, then let B' be a Δ_k^ℓ-optimal set of the same size with $m' = |\Delta_k^\ell(B')| < |\Delta_k^\ell(B)|$. Since the function $\Lambda_k^\ell(m)$ is nondecreasing, then

$$\Lambda_k^\ell(m-1) \geq \Lambda_k^\ell(m') \geq |B'| = |B| > \Lambda_k^\ell(m-1).$$

The last inequality holds because $\Lambda_k^\ell(A_{m-1})$ is a proper subset of B by the construction. Applying these arguments for all m satisfying (1) one gets nested Δ_k^ℓ-optimal subsets B_p for $p = 0, \ldots, |\mathcal{L}_\ell|$. $\square$

Due to this Lemma we will be concerned with Λ_k^ℓ-optimal sets only. Our main Theorem 2 now can be reformulated as follows.

Theorem 3

(*i*) $|\Lambda_0^\ell(\mathcal{O}_0^1(m))| \geq |\Lambda_0^\ell(A)|$ *for any* $A \subseteq \mathcal{L}_0$, $|A| = m$ *and* $0 < \ell \leq n-1$.

(*ii*) $|\Lambda_k^{n-1}(\mathcal{O}_k^2(m))| \geq |\Lambda_k^{n-1}(A)|$ *for any* $A \subseteq \mathcal{L}_k$, $|A| = m$ *and* $0 \leq k < n-1$.

In the proof of Theorem 3(*ii*) we will use the following assertion:

Remark 1 *For any $\ell > 0$ the subset $\Lambda_0^\ell(\mathcal{O}_0^1(m))$, $m = 1, \ldots, 2^{n+1} - 1$, is an initial segment of $\mathcal{L}_\ell$ ordered by $\mathcal{O}_\ell^1$.*

This immediately follows from the definitions of the order $\mathcal{O}^1$ and the function $\Lambda_0^\ell(\cdot)$.

3 Proof of Theorem 3

Let z be a point of $\mathrm{PG}\,(n,2)$. Consider the lines passing through z. Taking into account that each line consists of three points, denote for $x \in \mathrm{PG}\,(n,2)$ by $p_z(x)$ the third point on the line passing through z and x if $z \neq x$, and $p_z(z) = z$. Let S be some fixed set of points of $\mathrm{PG}\,(n,2)$. For $A \subseteq \mathcal{L}_0$ we introduce the projections

$$
\begin{aligned}
C_{z,S}(x) &= \begin{cases} p_z(x) & \text{if } x \in A \setminus S \text{ and } p_z(x) \in S \setminus A \\ x & \text{otherwise} \end{cases} \\
C_{z,S}(A) &= \{C_{z,S}(x) \mid x \in A\}.
\end{aligned}
$$

Lemma 2 *Let $A \subseteq \mathcal{L}_0$ and let S be a hyperplane. Then*

$$
|\Lambda_0^\ell(C_{z,S}(A))| \geq |\Lambda_0^\ell(A)| \tag{2}
$$

for any point $z \in \mathrm{PG}\,(n,2)$.

Proof.
To avoid trivial cases we assume $\Lambda_0^\ell(A) \neq \emptyset$, $z \notin S$ and $S \not\subseteq A$. Note that if T is an r-subspace for some r, $0 \leq r \leq n-1$, then $C_{z,S}(T)$ is a r-subspace of S. This is clear for $r = 0$ or if $z \in T$. If $r = 1$ and $z \notin T$, then the assertion follows from the Pasch axiom: a line which meets two sides of a triangle also meets the third (cf. Fig. 2a for the triangle $\{a, z, b\}$ and the line $\{x, y, c\}$). In this case if a line $\{x, y, c\}$ is not in S, then one of its points, say c, must be in S. Now the points $a = p_z(x)$ and $b = p_z(y)$ are in S and lie on a line passing through c. These arguments applied to any line in T imply the assertion for $r > 1$.

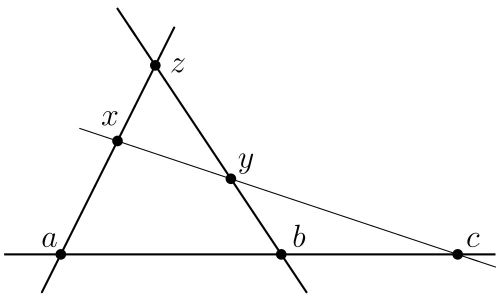

Figure 2: The Pasch axiom.

First consider the case $\ell = n-1$. In order to prove (2) we construct an injection $\theta : \Lambda_0^{n-1}(A) \mapsto \Lambda_0^{n-1}(C_{z,S}(A))$. Let $H \in \Lambda_0^{n-1}(A)$. If $z \in H$, then $C_{z,S}(H) = H$ and we put $\theta(H) = H$. Hence, we limit our attention to the hyperplanes from $\Lambda_0^{n-1}(A)$ not containing z. Let J be such a hyperplane. Then $C_{z,S}(J) = S$ and we put $\theta(J) = S$. For any other hyperplane $I \in \Lambda_0^{n-1}(A)$ with $z \notin I$, put

$$
\theta(I) = G := I \oplus J \oplus S,
$$

i.e. G is a set, each point of which satisfies the modulo 2 sum of equations for the hyperplanes I, J and S. Note that this definition also works for J. Obviously, G is a hyperplane and, thus, θ is injective. It remains to show that $G \subseteq C_{z,S}(A)$.

For that consider $x \in G$. Now x is contained in exactly one or in all three of I, J and S. If it is contained in all three, then also in $A \cap S$ and therefore in $C_{z,S}(A)$. If $x \in S \setminus A$, then for $y = p_z(x)$ it holds $y \in J \cap I$. Thus, $y \in A$ and $x = p_z(y)$. If finally x is contained in say J, but not in I and S, then $p_z(x) \in I$, since I contains a point of the line $\{z, x, p_z(x)\}$ and $z \notin I$. Hence $p_z(x) \in A$ and $x = C_{z,S}(x) \in C_{z,S}(A)$. This completes the proof of (2) for $\ell = n - 1$.

Now let $\ell < n - 1$. Furthermore, let R be an arbitrary ℓ-subspace of S, and let Q_R be the $(\ell + 1)$-subspace formed by the points of R and z. Denote by $A_\ell(z)$ the set of all ℓ-subspaces in $\Lambda_0^\ell(A)$ which contain the point z. One has

$$C_{z,S}(A) = \bigcup_{R \subset S} \tilde{C}_{z,R}(A \cap Q_R) \tag{3}$$

$$|\Lambda_0^\ell(A)| = \sum_{R \subset S} |\tilde{\Lambda}_0^\ell(A \cap Q_R)| - (2^{n-\ell} - 2) \cdot |A_\ell(z)|, \tag{4}$$

where the operators $\tilde{C}$ and $\tilde{\Lambda}$ at the right hand sides of (3) and (4) are applied to the subspace Q_R, in which R is a hyperplane, and the unions are taken over all the ℓ-subspaces R of S.

Since (3) is obvious, we show (4) only. For that consider an ℓ-subspace $T \in \Lambda_0^\ell(A)$. First we show that if $T \notin A_\ell(z)$, then T is contained just in one of the Q's. It is obvious if T is in S, since in this case T is one of the R's. Otherwise, if T is not in S, then $C_{z,S}(T)$ is an ℓ-subspace of S, and, thus, is one of the R's. Therefore, in both cases the corresponding subspace Q which contains T is defined uniquely, and, thus, T is counted exactly once in the sum (4).

On the other hand, if $T \in A_\ell(z)$, then T meets S in an $(\ell-1)$-subspace, which is contained in $2^{n-\ell} - 1$ of the R's and, thus, in so many of the Q's. Hence, in this case T is counted exactly $2^{n-\ell} - 1$ times in the sum (4).

An equality similar to (4) is also valid with respect to the set $B = C_{z,S}(A)$. Note that $B_\ell(z) = A_\ell(z)$. According to the arguments above $|\tilde{\Lambda}_0^\ell(B \cap Q_R)| \geq |\tilde{\Lambda}_0^\ell(A \cap Q_R)|$, which implies (2), and we have the lemma. □

Proof of Theorem 3(i).

We show by induction on n that there exists a sequence of projections of the form $C_{z,S}$ with respect to some appropriate points z and hyperplanes S, transforming any m-set $A \subseteq \mathcal{L}_0$ into $\mathcal{L}_0^1(m)$ without decreasing $|\Lambda_0^\ell(A)|$. For $n = 1, 2$ the Theorem is trivial. Let us make the inductive step for $n \geq 3$.

Consider first the hyperplane H_1 defined by $\beta_1 = 0$. If $A \subseteq H_1$, then the Theorem follows from the inductive hypothesis. Otherwise we apply the projection C_{z,H_1} for all points $z \in \mathrm{PG}\,(n, 2) \setminus H_1$ in succession. Notice that, $p_z(x) \in H_1$ for any point $x \in A \setminus H_1$. Since for each projection C_{z,H_1} holds $\sum_{x\in A} \text{lex}(x) \geq \sum_{x\in C_{z,H_1}(A)} \text{lex}(x)$, a finite number of the projections results in a set B with $C_{z,H_1}(B) = B$ for any $z \in \text{PG}\,(n,2) \setminus H_1$. Using Lemma 2 one has $|\Lambda_0^\ell(B)| \geq |\Lambda_0^\ell(A)|$. Clearly, either $B \subseteq H_1$ or $B \supset H_1$. In the first case the Theorem follows from the inductive hypothesis. Let us consider the second case.

In this case the set B consists of the hyperplane H_1 and of some other $m_1 = m - (2^n - 1)$ points outside H_1. Now let us consider the hyperplane H_2 defined by $\beta_2 = 0$. Our goal is to fill the hyperplane H_2 with the remaining m_1 points in such a way that the hyperplane H_1 is still in the resulting set too. To achieve this it is sufficient to apply the projection C_{z,H_2} for points z in $H_1 \setminus H_2$ only. Indeed, for each $x \in \text{PG}\,(n,2) \setminus (H_1 \cup H_2)$ and $y \in H_2 \setminus H_1$ there exists a point $z \in H_1 \setminus H_2$ such that $p_z(y) = x$. Therefore, applying to B the projections C_{z,H_2} for each point $z \in H_1 \setminus H_2$ in succession, one gets in the same way as before a set D with $|\Lambda_0^\ell(D)| \geq |\Lambda_0^\ell(B)|$.

Now, if $H_1 \cup H_2 \not\supseteq D$, then we turn to the hyperplane H_3 defined by $\beta_3 = 0$ and so on with hyperplanes H_j defined by $\beta_j = 0$, $j = 1, \ldots, n+1$. In the j^{th} step of this process ($j \geq 3$) we apply the projection C_{z,H_j} with each point $z \in (\bigcap_{i=1}^{j-1} H_i) \setminus H_j$. It is easily seen that for each $x \in \text{PG}\,(n,2) \setminus \bigcup_{i=1}^{j} H_i$ and $y \in H_j \setminus \bigcup_{i=1}^{j} H_i$ there exists a point $z \in (\bigcap_{i=1}^{j-1} H_i) \setminus H_j$ such that $p_z(y) = x$.

If we get as the result a set E with $E = \bigcup_{i=1}^{r} H_i$, then the Theorem is true since $E = \mathcal{O}_0^1(m)$. Otherwise

$$\bigcup_{i=1}^{r} H_i \subset E \subset \bigcup_{i=1}^{r+1} H_i,$$

for some $r < n$ (we assume that $m < |\mathcal{L}_0|$).

Consider the set $P = \bigcup_{i=1}^{r} H_i$. Clearly, $P = \mathcal{O}_0^1(|P|)$ and the set $P \cap H_{r+1}$ forms an initial segment of the order $\mathcal{O}_0^1$ in the hyperplane H_{r+1}. On the other hand, if Q is any initial segment of the order $\mathcal{O}_0^1$ in the hyperplane H_{r+1}, then for the set $R = P \cup Q$ it holds $R = \mathcal{O}_0^1(|R|)$.

Now we apply the inductive hypothesis to the set $E \cap H_{r+1}$ and transform it into an initial segment F of the order $\mathcal{O}_0^1$ in the hyperplane H_{r+1}. Taking into account the above remarks and the inclusion $P \cap H_{r+1} \subseteq F$, one has

$$|\Lambda_0^\ell(E)| = |\Lambda_0^\ell(P) \cup \Lambda_0^\ell(E \cap H_{r+1})| \leq |\Lambda_0^\ell(P) \cup \Lambda_0^\ell(F)| = |\Lambda_0^\ell(O_0^1(m))|.$$

Proof of Theorem 3(ii).

First consider the geometry $\text{PG}\,(n,2)$ and let $A \subseteq \mathcal{L}_k$ be Λ_k^{n-1}-optimal, with $|A| = m$. Denote $B = \mathcal{L}_{n-1} \setminus \Lambda_k^{n-1}(A)$ (cf. Fig. 3(a)).

Now consider the dual geometry $\text{PG}\,'(n,2)$. The set B corresponds to the set $B' = \phi(B) \subseteq \mathcal{L}_0'$ and we denote $D' = \Lambda_0^{n-1-k}(B')$ in $\mathcal{L}'$. One has

$$|\mathcal{L}_{n-1-k}' \setminus D'| \leq |A|. \tag{5}$$

Further denote $\tilde{B}' = \mathcal{O}_0^1(|B'|)$ and $\tilde{D}' = \Lambda_0^{n-1-k}(B')$ in $\mathcal{L}'$ (cf. Fig. 3(b)). Applying Theorem 3(i) to the set B' in PG$'(n,2)$ one has $|\tilde{D}'| \geq |D'|$. Therefore, (5) implies

$$|\mathcal{L}'_{n-1-k} \setminus \tilde{D}'| \leq |A|. \tag{6}$$

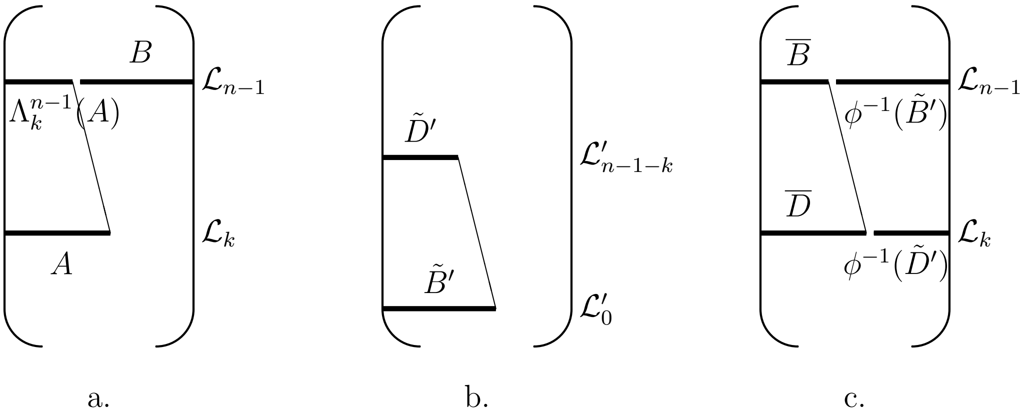

Figure 3: Usage of the duality in the proof of Theorem 3(ii).

Now return back to the geometry PG $(n,2)$. Since by Remark 1 the subsets $\tilde{B}'$ and $\tilde{D}'$ are initial segments of the orders $\mathcal{O}_0^1$ and $\mathcal{O}_{n-1-k}^1$ in $\mathcal{L}'$ respectively, the sets $\overline{B} = \mathcal{L}_{n-1} \setminus \phi^{-1}(\tilde{B}')$ and $\overline{D} = \mathcal{L}_k \setminus \phi^{-1}(\tilde{D}')$ are initial segments of the orders $\mathcal{O}_{n-1}^2$ and $\mathcal{O}_k^2$ in $\mathcal{L}$ respectively (cf. Fig. 3(c)). Since ϕ is a bijection, similarly to (5) one has

$$\overline{B} \subseteq \Lambda_k^{n-1}(\overline{D}). \tag{7}$$

Since $\overline{D} \subseteq \mathcal{O}_k^2(m)$ by (6), applying (7) we get

$$|\Lambda_k^{n-1}(\mathcal{O}_k^2(m))| \geq |\Lambda_k^{n-1}(\overline{D})| \geq |\overline{B}| = |\mathcal{L}_{n-1} \setminus B| = \Lambda_k^{n-1}(m),$$

which shows that the set $\mathcal{O}_k^2(m)$ is Λ_k^{n-1}-optimal. □

4 The general case

First note that the order $\mathcal{O}^1$ does not work for maximization of $\Lambda_k^l(\cdot)$ for k,l with $0 < k < l < n$. Indeed, consider the case $n = 3$ and let $A = \mathcal{O}_1^1(18) \subseteq \mathcal{L}_1$. One has $|\Lambda_1^2(A)| = 2$. However, A is not Λ_1^2-optimal, since for the following set $B \subseteq \mathcal{L}_1$ defined by the elements numbered in the order $\mathcal{O}_1^1$:

$$B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 22\}$$

one has $|B| = |A| = 18$ but $|\Lambda_1^2(B)| = 3$.

On the other hand the order $\mathcal{O}^2$ is not quite good for maximization of $\Lambda_0^1(\cdot)$, because in the case $n = 3$, for example, the set $\Lambda_0^1(\mathcal{O}_0^2(6))$ is not an initial segment in $\mathcal{L}_1$ with respect to this order. This property is very important in the light of the Macaulay posets introduced below.

Now we return back to the original statement of the problem, namely to the minimization of the shadow. We strengthen the notion of nestedness in the shadow minimization problem by introducing an important class of Macaulay posets (cf. [1]). Let P be a ranked poset and denote by P_ℓ the set of elements of P of rank ℓ. Similarly as above we can define the shadow of a subset of P_ℓ by using the partial order on P. A poset P is called Macaulay if there exists a total order $\mathcal{Q}$ of elements of P, such that for any k, ℓ $(k < \ell)$ and m the initial segment of length m of the induced order $\mathcal{Q}_\ell$ on P_ℓ has minimal shadow in P_k, and this shadow itself is an initial segment of the induced order $\mathcal{Q}_k$.

Macaulay posets have many helpful properties which provide solutions for a number of related extremal problems (see [1] for more details). Clearly, for $n = 2$ the poset $\mathcal{L}$ is Macaulay (cf. Fig. 1).

Remark 2 *The poset $\mathcal{L}$ is not Macaulay for $n \geq 3$.*

Indeed, let $n \geq 3$ and consider just the three bottom levels of $\mathcal{L}$. Assume that there exist the orders $\mathcal{Q}_i$, $i = 0, 1, 2$ of points, lines and planes of PG $(n, 2)$ with the Macaulay property. We will show that this leads to a contradiction. If a subspace a of PG $(n, 2)$ contains a subspace b, we say that a covers b, or b is covered by a.

Denote $A = \mathcal{Q}_2(3)$. Since A covers the minimal number of points, Theorem 2(i) implies that this number is 13. Let us compute the number of lines covered by A. Since each plane in PG $(n, 2)$ contains 7 lines and each two planes have at most one line in common, then the total number of lines in 3 planes is at least $3 \cdot 7 - 3 = 18$. On the other hand, the set $\mathcal{O}_2^2(3)$ covers exactly 18 lines.

Therefore, A covers exactly 18 lines and 13 points. According to the same arguments, the configuration $B = \mathcal{Q}_2(2)$ covers 13 lines and 11 points. Hence, the plane $a = A \setminus B$ covers 5 new lines $b_1, \ldots, b_5$ and 2 new points x_1, x_2, covered by these 5 lines.

Consider now the set $I = \mathcal{Q}_1(17)$. Then, following our assumption $\Delta_1^2(B) \subset I \subset \Delta_1^2(A)$ and I should be an optimal subset. Therefore, $|\Delta_0^1(I)| = 12$. Without loss of generality we can assume $I = \Delta_1^2(B) \cup \{b_1, b_2, b_3, b_4\}$ and $\Delta_0^2(I) = \Delta_0^2(B) \cup x_1$.

Further notice that for the point $x_2 \in a$ there exist exactly 4 lines of a, which do not cover it (that holds for any point of a plain). In our denotations these lines are just $b_1, \ldots, b_4$. Since the plane a consists of 7 lines and $|\Delta_1^2(A) \setminus \Delta_1^2(B)| = 5$, there must be at least 2 lines in $\Delta_1^2(B)$ which cover x_2. In other words, the configuration B of planes covers the point x_2, which is a contradiction. $\square$

Finally, let us mention an interesting phenomena in the case $n = 3$. In this case the order $\mathcal{O}^2$ works for minimization of $\Delta_1^2(\cdot)$ and the order $\mathcal{O}^1$ works for minimization of $\Delta_0^1(\cdot)$.

Moreover, both orders work for minimization of $\Delta_0^2(\cdot)$. However, as Remark 2 shows, there is no universal order, which would provide the Macauleyness of the poset $\mathcal{L}$.

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