Isoperimetric Problems in Discrete Spaces*

Sergei L. Bezrukov

Abstract

This paper is a survey on discrete isoperimetric type problems. We present here as some known facts about their solutions as well some new results and demonstrate a general techniques used in this area. The main attention is paid to the unit cube and cube like structures. Besides some applications of the isoperimetric approach are listed too.

1 Introduction

This paper is devoted to the discrete isoperimetric problem. This problem may be considered as an analog of the well known continuous problem and has some similar features. The discrete isoperimetric problem began to be studied a very long ago and a lot about it's solutions is known now. If there is the only solution of the continuous version, the discrete one, considered for the unit cube, has generally more solutions with much more rich structure, which have no direct continuous analogs. It is mainly due to the facts that, at first not for all values of cardinality of a subset (which is defined as the number of cube points in the subset) there is a discrete ball of this cardinality. At the second, if d is an odd number then there is no a ball in the n-cube with diameter d. These two facts (and some others which are not so clear) create a lot of difficulties for specification of all solutions of the discrete isoperimetric problem and make us consider it not as a topological problem but rather as a pure combinatorial one.

As it was mentioned above, our problem has a lot of nonisomorphic solutions, so it is of interest to study them. We present here a collection of some results in this direction, and demonstrate the proof techniques. The paper is organized as follows. Sections 2–5 are devoted to the n-dimensional unit cube and the vertex version of the isoperimetric problem. Other structures are considered briefly in Section 7. The next Section contains some necessary basic definitions and presents three different approaches for the proof of the Main Isoperimetric Theorem (which shows how to construct one solution), namely the structural transformations, shifting techniques and a numerical method. We present there the sketches of proofs of this Theorem.

A very important definition of critical subset and critical cardinality is introduced in Section 3. There one can find a complete specification of all subsets of the n-cube with

*Published in: *Conference on Extremal Problems for Finite Sets*, 1991, Visegrad, Hungary, Extended Abstract, 14–36 and *Extremal Problems for Finite Sets*. J. Bolyai Soc. Math. Stud. 3, P. Frankl, Z. Füredi, G. Katona, D. Miklos eds., Budapest 1994, 59–91.

such cardinality and some reduction theorems. The importance of these notions is due to the fact that the number of critical cardinalities of the n-cube equals 2^{n-1}, and so for a half of the possible values of cardinality we know all the solutions. In fact we are also able to specify all the solutions for an exponentially large number of other cardinalities, which is shown in Section 4.

The results of Sections 2–4 are already known and are based on [3–6, 13, 16]. In Section 5 we show some difficulties on the way of structural specification of all the solutions and propose an approach for an iterative specification. Besides, this Section contains a classification of solutions and states some open problems.

Section 6 contains known results about edge version of the isoperimetric problem. Finally, Section 7 is devoted to various versions of the vertex isoperimetric problem and to applications of the listed results for the solution of some related problems.

2 The Main Isoperimetric Theorem

Denote by B^n the n-dimensional unit cube, i.e. the collection of all binary n-dimensional vectors, and let ρ be the Hamming metric.

Definition 2.1 *A point $\alpha \in A \subseteq B^n$ is called the inner point of a set A if $S_1^n(\alpha) \subseteq A$, where $S_r^n(\alpha)$ denotes the n-dimensional ball of radius r centered in α. In opposite case we call α the boundary point of A.*

Denote by $P(A)$ ($\Gamma(A)$) the collection of all inner (boundary) points of A.

Definition 2.2 *A set $A \subseteq B^n$ is called optimal if $|\Gamma(A)| \leq |\Gamma(B)|$ (or, which is the same, $|P(A)| \geq |P(B)|$) for any $B \subseteq B^n$, $|B| = |A|$.*

This Section is devoted to the proof of optimality of the set L_m^n, called standard. L_m^n is determined as the initial segment of length m of the following order $\mathcal{L}$ of vertices of B^n. We say that $\alpha \in B^n$ precedes $\beta \in B^n$ in order $\mathcal{L}$ iff

1. $\|\alpha\| < \|\beta\|$, or
2. $\|\alpha\| = \|\beta\|$, and β precedes α lexicographically.

Here as well as in the whole paper we denote by $\|\alpha\|$ the sum of it's entries. The optimality of L_m^n was first proved by Harper [38]. Although his proof is very hard and incomplete, it contains almost all elements, which later reduced to the creation of modern powerful techniques. In this Section we present three different (simple) methods to prove the following result, which we call The Main Isoperimetric Theorem.

Theorem 2.1 *L_m^n is an optimal set for all n and m, $1 \leq m \leq 2^n$.*

This is the structural version of the isoperimetric theorem, deals with the structure of solutions. Of course, another kinds of statements are possible, and mainly the isoperimetric inequalities, i.e. all possible inequalities between the size of A and it's boundary. The first such inequality was derived in [54] and states the following.

Theorem 2.2 [54]. *If* $|A| = \sum_{i=0}^{k}\binom{n}{i} + \delta$, $0 \le \delta < \binom{n}{k+1}$, *then* $|\Gamma(A)| \ge \binom{n}{k} + \delta\left(1 - \frac{k+1}{n-k}\right)$.

This Theorem implies that if $|A|$ is a spherical cardinality, i.e. $\delta = 0$ and $|A|$ equals the cardinality of a ball of radius k, then the mentioned ball is a solution of the isoperimetric problem, and so the inequality in the statement becomes an equality. Moreover, this inequality has a simple interpretation. As we know now, the function $|\Gamma(A)|$ increases from $\binom{n}{k}$ to $\binom{n}{k+1}$ as $|A|$ increases from $\sum_{i=0}^{k}\binom{n}{i}$ to $\sum_{i=0}^{k+1}\binom{n}{i}$. And the inequality of Theorem 2.2 means that the function $|\Gamma(A)|$ grows faster than linearly, since the right hand side is the linear interpolation of this function.

Isoperimetric inequalities are very useful for applications. A lot of isoperimetric inequalities for different structures one can find in [22–28].

We will assume in sections 2–7 that m denotes the cardinality of A and that as well as in Theorem 2.2 it is represented in the following canonical form

$$|A| = \sum_{i=0}^{k}\binom{n}{i} + \delta, \quad 0 \le \delta < \binom{n}{k+1}.$$

It is clear that the numbers k and δ are determined uniquely by m, and we will use them without any references to the canonical representation.

For a point $\alpha \in B^n$ and integer i, $1 \le i \le n$ we denote by $\pi_i(\alpha)$ the point which is obtained from α by inverting it's i-th entry. If $A \subseteq B^n$ and $\sigma \in \{0,1\}$, then let

$$\begin{aligned}
\pi_i(A) &= \{\pi_i(\alpha) : \alpha \in A\}, \\
A^\sigma(i) &= \{\alpha = (\alpha_1, \ldots, \alpha_n) \in A : \alpha_i = \sigma\}, \\
m_0 &= |A^0(i)|, \quad m_1 = |A^1(i)|, \\
p(m,n) &= |P(L_m^n)|.
\end{aligned}$$

a. Structural Transformations.

The main idea and a sketch of the proof is due to Kleitman [43] (see also [6]). This techniques was successfully used also by different authors and applied to a lot of problems. We say that a set A is i-compressed if $A^0(i)$ and $A^1(i)$ are standard sets in $(n-1)$ dimensions. For any subset $A \subseteq B^n$ and a fixed i we may construct the i-compressed set from A by making $A^0(i)$ and $A^1(i)$ standard sets in $(n-1)$ dimensions. We denote this set by C_iA.

Lemma 2.1 $|\Gamma(C_iA)| \le |\Gamma(A)|$.

Proof.
We use induction on n. There is no loss of generality to assume $|A^0(i)| \ge |A^1(i)|$. Then

$$|\Gamma(A)| \ge \max\{|\Gamma(A^0(i))| + |\Gamma(A^1(i))|, |\Gamma(A^1(i))| + |A^0(i) \setminus \pi_i(A^1(i))|\}$$

(the operator Γ in the right hand side applies in $(n-1)$ dimensions). By induction the right hand side is not smaller than

$$\max\{|\Gamma(C_iA^0(i))| + |\Gamma(C_iA^1(i))|, |\Gamma(C_iA^1(i))| + |A^0(i)| - |A^1(i)|\},$$

which equals $|\Gamma(C_iA)|$.

So, the compression does not increase the size of the boundary. Notice that $\mathrm{LEX}(C_iA) \leq \mathrm{LEX}(A)$, where $\mathrm{LEX}(A) = \sum_{\alpha \in A} \mathrm{LEX}(\alpha)$ and for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\mathrm{LEX}(\alpha) = \sum_{i=1}^{n} \alpha_i 2^{n-i}$. Therefore one can continue the compression for $i = 1, \ldots, n$, then again for $i = 1, \ldots, n$ and so on until he obtains a set B with $C_iB = B$ for $1 \leq i \leq n$. We call such a set B compressed.

Lemma 2.2 [3,6]. *If B is a compressed set then either $B = L^n_m$ or $m = 2^{n-1}$ and $B = L^n_{2^{n-1}-1} \cup \alpha_{2^{n-1}+1}$, where $\alpha_{2^{n-1}+1}$ is the $(2^{n-1}+1)$-st point in order $\mathcal{L}$.*

To prove this Lemma one should consider the lexicographically greatest point α in B and the lexicographically least point β in the complement of B, and check $\beta \in B$ with the only exclusion listed in the statement.

Now to complete the proof of Theorem 2.1, one should apply Lemma 2.1 and Lemma 2.2 and notice that the second set defined in Lemma 2.1 has a greater boundary than the first one. ∎

b. Shifting Techniques.

The second method uses some structural transformations as well as properties of binomial coefficients. The proof is due to Frankl [34] and is very similar to his short proof of the Kruskal-Katona theorem [33]. We use here this method only to prove the weaker version of Theorem 2.1. Denote

$$\Theta(A) = \{\alpha \in B^n : \rho(\alpha, A) \leq 1\} .$$

Since $\Theta(A) \setminus A = \Gamma(B^n \setminus A)$, then minimization of Γ is equivalent to minimization of Θ.

Theorem 2.3 [34]. *Let $A \subseteq B^n$ and*

$$|A| \geq \binom{n}{n} + \cdots + \binom{n}{k+1} + \binom{x}{k},$$

for some real x, $k \leq x \leq n$. Then

$$|\Theta(A)| \geq \binom{n}{n} + \cdots + \binom{n}{k} + \binom{x}{k-1}.$$

Recall the definitions of the pushing-up operator T_i, $1 \leq i \leq n$, and of the shifting operator S_{ij}, $1 \leq i < j \leq n$ for $A \subseteq B^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in A$:

$$\begin{aligned} T_i(\alpha) &= \begin{cases} \alpha^* = \pi_i(\alpha), \text{ if } \alpha_i = 0 \text{ and } \alpha^* \notin A \\ \alpha, \text{ otherwise} \end{cases}, \\ S_{ij}(\alpha) &= \begin{cases} \alpha^* = \pi_j(\pi_i(\alpha)), \text{ if } \alpha_i = 0,\ \alpha_j = 1 \text{ and } \alpha^* \notin A \\ \alpha, \text{ otherwise} \end{cases}, \end{aligned}$$

and let $T_i(A) = \{T_i(\alpha) : \alpha \in A\}$ and $S_{ij}(A) = \{S_{ij}(\alpha) : \alpha \in A\}$.

Lemma 2.3 [34]. *$\Theta(T_i(A)) \subseteq T_i(\Theta(A))$ for any $A \subseteq B^n$.*

So, we may assume that $T_i(A) = A$ for $1 \le i \le n$, and then

Lemma 2.4 [34]. *$\Theta(S_{ij}(A)) \subseteq S_{ij}(\Theta(A))$ holds for all $1 \le i < j \le n$.*

Therefore, we additionally assume that $S_{ij}(A) = A$ holds.

Lemma 2.5 [34].
(i) $|\Theta(A)| = |A^1(1)| + |\Theta(A^1(1))|$,
(ii) $\Theta(A^0(1)) \subseteq \pi_1(A^1(1))$.

In view of Lemmata 2.3 and 2.4 we may assume that A is pushed up and shifted. Apply now induction on n, the case $n = 1$ being trivial.

(a) Assume $$|A^1(1)| \ge \binom{n-1}{n-1} + \cdots + \binom{n-1}{k} + \binom{x-1}{k-1}.$$

By the inductive hypothesis we have

$$|\Theta\left(A^1(1)\right)| \ge \binom{n-1}{n-1} + \cdots + \binom{n-1}{k-1} + \binom{x-1}{k-2}$$

and Theorem 2.2 follows from Lemma 2.5(i).

(b) Assume $$|A^1(1)| < \binom{n-1}{n-1} + \cdots + \binom{n-1}{k} + \binom{x-1}{k-1}.$$

Now $$|A^0(1)| > \binom{n-1}{n-1} + \cdots + \binom{n-1}{k+1} + \binom{x-1}{k},$$

thus by the inductive hypothesis

$$|\Theta\left(A^0(1)\right)| \ge \binom{n-1}{n-1} + \cdots + \binom{n-1}{k} + \binom{x-1}{k-1}$$

follows, and by Lemma 2.5 *(ii)* we get a contradiction

$$|A^1(1)| \ge \binom{n-1}{n-1} + \cdots + \binom{n-1}{k} + \binom{x-1}{k-1}.$$

∎

The same method may be successfully applied for the proof for the strong numerical analog of Theorem 2.1 (see Theorem 2.3 below).

c. Numerical Computations.

The third also numerical approach is due to Katona [41]. It seems quite similar to the second one, but in contradistinction to it does not claim any geometrical arguments. All the arguments are very formal and pure arithmetical. Of course a qualified reader can see a geometrical interpretation of them.

It is easy to show that there is a unique representation of m in the form

$$m = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{k+1} + \binom{a_k}{k} + \cdots + \binom{a_t}{t}$$

$$(n > a_k > a_{k-1} > \ldots > a_t \geq t \geq 1)\,.$$

Being based on this representation let us introduce the function

$$G(n,m) = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{k+1} + \binom{n}{k} + \binom{a_k}{k-1} + \cdots + \binom{a_t}{t-1}$$

if $m > 0$ and $G(n,0) = 0$.

Theorem 2.4 [41]. $|\Theta(A)| \geq G(n, |A|)$.

It is really an analog of Theorem 2.1 since $|\Theta(L^n_m)| = G(n,m)$.

Lemma 2.6 *If* $0 \leq m_0 \leq m_1$ *then*

$$G(n, m_0 + m_1) \leq max(m_1, G(n-1, m_0)) + G(n-1, m_1).$$

Let i be an arbitrary integer, $1 \leq i \leq n$. Without loss of generality we assume $m_0 \leq m_1$.

Case 1. $m_1 \leq G(n-1, m_0)$. By the inductive hypothesis one has

$$|\Theta(A^1(i))| \geq G(n-1, m_1) \qquad \text{and} \qquad |\Theta(A^0(i))| \geq G(n-1, m_0)$$

Hence, $|\Theta(A)| \geq |\Theta(A^0(i))| + |\Theta(A^1(i))| \geq G(n-1, m_0) + G(n-1, m_1)$, which is at least $G(n, m_0 + m_1) = G(n, m)$ by Lemma 2.6.

Case 2. $G(n-1, m_0) \leq m_1$. Since

$$|\Theta(A)| \geq m_1 + |\Theta(A^1(i)|,$$

one has by induction

$$|\Theta(A)| \geq m_1 + G(n-1, m_1),$$

which by Lemma 2.6 is at least $G(n, m_0 + m_1) = G(n, m)$, and the proof is completed. ∎

3 General Properties of Optimal Subsets

Now we begin to investigate the collection of all optimal subsets. Let us set up at once which subsets we call different. Remind, that there are two mappings $B^n \mapsto B^n$ which do not change the distance between any two points of B^n. The first such transformation is shift on a fixed vector $\nu = (\nu_1, \ldots, \nu_n)$. It replaces each point $\alpha \in B^n$ to $\alpha \oplus \nu$, where $\oplus$ means the componentwise modulo 2 addition. The second transformation is reflection. If ζ denotes a permutation on a set $\{1, \ldots, n\}$, then the reflection replaces each point $\alpha \in B^n$ to $\zeta(\alpha)$. That means we permute entries of α in accordance with ζ.

Definition 3.1 *Let $A, B \subseteq B^n$. We call the sets A, B isomorphic if $A = \zeta(B) \oplus \nu$ for some permutation ζ and $\nu \in B^n$ and denote $A \equiv B$.*

Since $|\Gamma(A)| = |\Gamma(B)|$ for isomorphic sets and A and B are in fact the same sets, we consider optimal subsets up to isomorphism.

Definition 3.2 *A set $A \subseteq B^n$ is called i-normalized if $|A^0(i)| \geq |A^1(i)|$. If A is i-normalized for all $i = 1, \ldots, n$, we simply call it normalized.*

It is clear that without loss of generality we may consider only normalized subsets. Indeed, if $|A^0(i)| \leq |A^1(i)|$ for some i, then by replacing zeros to ones and vice versa in the i-th entries of vertices of A, we obtain an i-normalized set, which is isomorphic to the initial one. So we can always assume that A is a normalized subset. First we present two very useful propositions about intersection of an optimal sets with the n-cube hyperplanes. They give a tool for the proof by induction and will be used by us later in Section 5. Some more facts of this type one can find in [6]. The following theorem is a more general form of some propositions appeared in Section 2 (see also [54]).

Theorem 3.1 [6]. *Let A be an optimal normalized subset. Then for each i, $1 \leq i \leq n$, the following hold:*

a) if $p(m_0, n-1) < m_1$ then

$$\Gamma(A) = \Gamma(A^0(i)) \cup \Gamma(A^1(i)),$$

moreover $A^0(i)$ and $A^1(i)$ are optimal subsets in $(n-1)$ dimensions, $\pi_i(P(A^0(i))) \subset A^1(i)$, $\pi_i(P(A^1(i))) \subset A^0(i)$;

b) if $p(m_0, n-1) > m_1$ then

$$\Gamma(A) = (A^0(i) \setminus \pi_i(A^1(i))) \cup \Gamma(A^1(i)),$$

moreover $A^1(i)$ is optimal subset and $\pi_i(P(A^0(i))) \supseteq A^1(i)$;

c) if $p(m_0, n-1) = m_1$ then

$$\Gamma(A) = \Gamma(A^0(i)) \cup \Gamma(A^1(i)) = (A^0(i) \setminus \pi_i(A^1(i))) \cup \Gamma(A^1(i)),$$

moreover $A^0(i)$ and $A^1(i)$ are optimal subsets in $(n-1)$ dimensions, $P(A^0(i)) = \pi_i(A^1(i))$.

So, as it is shown, there are in fact two formulae to express the boundary of the whole set being based on the boundaries of it's sections. Now we will see that one can always use the formula from the part a) of the theorem with one very special exception.

Theorem 3.2 [6]. *For any normalized optimal set $A \subseteq B^n$ either there exists an i for which $|\Gamma(A)| = |\Gamma(A^0(i))| + |\Gamma(A^1(i))|$, or $A = S^n_{n-1}(0, \ldots, 0)$.*

The next important theorem shows in particular a similarity with the continuous isoperimetric problem.

Theorem 3.3 [6]. *For each optimal set $A \subseteq B^n$ there exists a point $\alpha \in A$ such that*

$$S_k^n(\alpha) \subseteq A.$$

It is clear that there is no ball of radius $k+1$ in A, hence by Theorem 3.3 a ball of the maximal possible radius must be in A. If m is a spherical cardinality (i.e. $\delta = 0$), then a ball is the unique solution of the isoperimetric problem (up to location of it's center).

Therefore for a complete specification of the set of solutions it is sufficient to determine admissible positions of the rest δ points. The next theorem characterizes optimal subsets from another point of view.

Theorem 3.4 [16]. *If A is an optimal subset then $P(A)$ is the optimal subset too.*

Let us discuss some corollaries from this theorem. At first $P(P(\ldots P(A)\ldots)$ (l times) is the optimal subset for any $l = 1, \ldots, k$. Therefore any optimal set A contains a whole cascade of optimal subsets. Moreover, since $|P(P(\ldots P(A)\ldots)| = |P(P(\ldots P(L_m^n)\ldots)|$ (k times), one has $P(P(\ldots P(A)\ldots) \neq \emptyset$, which is equivalent to Theorem 3.3. In addition it gives a tool to determine the number of points α for which $S_k^n(\alpha) \subseteq A$ holds. This number equals $|P(P(\ldots P(L_m^n)\ldots)|$ (k times) and may be computed by the well-known formula from [41].

4 Critical Subsets and Critical Cardinalities

In this Section we present some more cases when all the optimal subsets are known. Of course the situation hardly depends on m (or more strictly on δ).

Definition 4.1 *A point $\alpha \in A$ is called the free point of A if $S_1^n(\alpha) \cap P(A) = \emptyset$.*

Denote by $S(A)$ the collection of all free points of a set A. It is clear that if $\alpha \in S(A)$ then $P(A \setminus \alpha) = P(A)$, and so if A is optimal then $A \setminus \alpha$ is optimal too. Moreover, if we now consider the set $A \setminus \alpha$ and add to it an arbitrary point form it's complement, then we also get an optimal m-element set B with the same collection of inner points. Indeed, $|P(B)| \geq |P(A)|$ since $p(m,n)$ is the nondecreasing function on m. On the other hand the strict inequality contradicts to the optimality of A. Of course the same holds if we delete and then add more free points of A. These arguments lie in a basis of a more general construction but first we need the following definition.

Definition 4.2 *A set $A \subseteq B^n$ is called critical if $S(A) = \emptyset$. We assume that $\emptyset$ is the critical set too.*

A simple example of a critical set is a ball of nonzero radius. The significance of these two definitions follows from the reduction of specification of noncritical subsets to specification of critical one.

Definition 4.3 *A number m is called the critical cardinality if L_m^n is a critical set.*

The following construction shows the induction. Denote by m_0^* the maximal critical cardinality, less or equal m, i.e. $m_0^* = m - |S(L_m^n)|$, and let $m_0^*, \dots, m_l^*$ $(m_0^* < m_1^* < \dots < m_l^* \le m)$ be all the cardinalities for which there exist optimal critical subsets of B^n. Consider the following families F_i, $i = 0, \dots, l$, of m-element subsets. In order to obtain the family F_i, we first construct all m_i^*-element optimal critical subsets, and then add arbitrary $m - m_i^*$ points to each of them. In such a way for every constructed m_i^*-element subset we include to F_i $\binom{2^n - m_i^*}{m - m_i^*}$ different m-element sets.

Theorem 4.1 [16]. *All subsets in $\{F_i,\ i = 0, \dots, l\}$ are pairwise different and each subset is optimal. Moreover, if A is an m-element optimal subset then $A \in F_i$ for some i.*

Therefore it is sufficient to consider critical optimal subsets only.

Lemma 4.1 [6]. *If L_m^n is a critical subset, then all the m-element optimal subsets are critical too.*

So there are some m for which all the m-element optimal subsets are critical. It is not remarkable since the function $p(m, n)$ is monotonically nondecreasing on m as we have mentioned, and the value of $p(m, n)$ changes as m passes a critical cardinality. It is known much more about the structure of an optimal subset if m is a critical cardinality. For example Theorem 3.3 can be strengthened by the following.

Theorem 4.2 [6]. *If $A \subseteq B^n$ is an optimal subset of a critical cardinality then there exists a point $\alpha \in A$ such that*

$$S_k^n(\alpha) \subseteq A \subseteq S_{k+2}^n(\alpha).$$

For a normalized set we may let $\alpha = (0, \dots, 0)$. Unfortunately the second inclusion of this theorem cannot be strengthened.

Theorem 4.3 [6]. *If A is a normalized set of a critical cardinality and*

$$S_k^n(0, \dots, 0) \subseteq A \subseteq S_{k+1}^n(0, \dots, 0)$$

holds, then $A \equiv L_m^n$.

Therefore for the complete characterization of all optimal subsets in the case when m is a critical cardinality it is sufficient to determine the way to distribute δ points by the two consecutive levels of the n-cube. The following theorem gives us a way for such characterization by induction.

Theorem 4.4 [6]. *For each optimal subset $A \subseteq B^n$ of a critical cardinality there exists a number i such that*

$$|A^0(i)| = \sum_{t=0}^{k} \binom{n-1}{t} + \left(\delta \triangle \binom{n-1}{k}\right), \quad |A^1(i)| = \sum_{t=0}^{k} \binom{n-1}{t} - \left(\binom{n-1}{k} \triangle \delta\right).$$

Moreover for such i propositions of cases a) or c) of theorem 3.1 hold. (here the symmetric difference of numbers equals the usual difference if it is positive and 0 otherwise)

Therefore both $A^0(i)$ and $A^1(i)$ are optimal subsets, and as it seen at least one of them has a spherical cardinality. By Theorem 3.3 this part is a ball. Moreover it is easy to show that the second part has a critical cardinality (in $n-1$ dimensions). So we may apply Theorem 4.4 to it again and obtain four $(n-2)$-dimensional subcubes. It is clear that at least in three of these subcubes, the corresponding parts of A are balls. If the forth part is not a ball, we can continue the division until all the parts are balls. That is our approach to specification.

Theorem 4.5 [13]. *If $A \subseteq B^n$ is a normalized subset of a critical cardinality then there exists a number $D(m)$ such that B^n may be divided into $(n-D(m))$-dimensional subcubes in such a way that all the parts of the set A in these subcubes are balls. Moreover the centers of all but one of these balls lie in the origins of the corresponding subcubes. The center of the last ball may lie either in the origin of the subcube or in some point of the first level of this subcube.*

How many steps do we need ?

Theorem 4.6 [13]. *Let m be a critical cardinality. Then*

a) there exist numbers $l_1, m_1, l_2, m_2, \ldots, l_r, m_r$, $(0 < l_1 \le m_1 < l_2 \le m_2 < \ldots < l_r \le m_r)$, such that δ may be uniquely represented in the form

$$\delta = \binom{n-l_1}{k-l_1+1} + \binom{n-l_1-1}{k-l_1+1} + \cdots + \binom{n-m_1}{k-l_1+1} +$$

$$\binom{n-l_2}{k-l_1-l_2+m_1+2} + \cdots + \binom{n-m_2}{k-l_1-l_2+m_1+2} +$$

$$\binom{n-l_r}{k-\sum_{i=1}^{r} l_i + \sum_{i=1}^{r-1} m_r + r} + \cdots + \binom{n-m_r}{k-\sum_{i=1}^{r} l_i + \sum_{i=1}^{r-1} m_r + r};$$

b) The number $D(m)$ of steps in the division process equals m_r.

Let for example $m = \sum_{t=0}^{k} \binom{n}{t} + \binom{n-1}{k}$. Then $D(m) = 1$ and by Theorem 4.5 there exists a number i such that an optimal normalized subset $A \in B^n$ has the following structure: $A^0(i) = S_k^{n-1}(0,\ldots,0)$ and $A^1(i)$ is either $S_k^{n-1}(0,\ldots,0)$ or $S_k^{n-1}(\alpha)$ for some point α of norm 1 (in $n-1$ dimensions).

The next proposition gives a criterion to determine whether m is a critical cardinality. For that purpose denote by $\alpha(L_m^n)$ the last point of L_m^n in order $\mathcal{L}$.

Lemma 4.2 [16]. *m is a critical cardinality (in n dimensions) iff the n-th entry of $\alpha(L_m^n)$ equals* 1.

Therefore, the number of critical cardinalities equals 2^{n-1} and we know the structure of all solutions in a lot of cases. Imagine now the situation when for a given m there exists the only nonempty class F_0, i.e. $l = 0$. Then by Theorem 4.2 we know all the the optimal subsets too. We call such a number m the good cardinality. The most pleasant fact is that there is a lot of good cardinalities.

Theorem 4.7 [16]. *If $p(m,n)$ is a critical cardinality then m is a good cardinality.*

So we know all the optimal subsets additionally at least in $\sum_{j=2}^{n-3}(j-1)2^{n-j-3} \sim 2^{n-3}$ cases. We can even increase this number if one should have a criterion to determine whether m is a good cardinality. Unfortunately it is very difficult to check it effectively. The only known step in this direction is given by the following theorem. But first we need to introduce some auxiliary isoperimetric type problem. For a given integer p consider the family $A_p = \{A \subseteq B^n : |P(A)| = p\}$. The new problem is to find a subset $A \in A_p$ with minimal boundary. Denote this value by $g(p,n)$.

This new problem and the original one have a lot in common. If for a given p there exists an optimal subset A with $|P(A)| = p$ then A is a solution of this new problem. So it is interesting only if for a given p there is no an optimal subset with $|P(A)| = p$. The minimal such p equals n since the function $p(m,n)$ has a jump from $n-1$ to $n+1$ as m increases from $\binom{n}{2}+n$ to $\binom{n}{2}+n+1$.

Theorem 4.8 [16]. *An m-element optimal critical subset exists iff*

$$m - p(m,n) = g(2^n - m, n).$$

Let us mention an open problem to build a p-optimal subset and/or to determine the number $g(p,n)$.

5 Iterative Generating of Optimal Subsets

In this Section we show why the problem of specifying of optimal subsets is so difficult and propose an approach for classification of optimal subsets and further research.

At first let us discuss what is a specification of all solutions. A good approach was used to construct one solution. Namely it was created a linear order $\mathcal{L}$ of the vertices of the n-cube, any initial segment of which induces a solution of the isoperimetric problem.

Definition 5.1 *A linear order of vertices of B^n is called optimal if any initial segment of it induces an optimal subset.*

It would be nice if we can find several optimal orderings $\mathcal{L}_1$,$\mathcal{L}_2$, …,$\mathcal{L}_l$ (l may be infinite) such that if A is an optimal set then either A is induced by $\mathcal{L}_1$ or … or by $\mathcal{L}_l$. Then it would be sufficient to describe the orders only to cover all the possible cases. But it is a wrong way.

Theorem 5.1 [16]. *If A is a critical subset induced by an optimal order then $|A|$ is a critical cardinality.*

But there are optimal critical subsets with noncritical cardinality. Therefore we cannot get much this way. Unfortunately it is not the only difficulty. In order to show the other one and simultaneously formulate our approach we need the following transformation which converts a subset $A \subseteq B^n$ into some subset of B^{n+1}. Notice first that a critical subset A may be considered as the union of balls of nonzero radii. For example $A =$

$\{\cup S_1^n(\alpha) : \alpha \in P(A)\}$. Of course such representation of A as the union is not unique, but let us fix some of them

$$A = S_{r_1}^n(\alpha_1) \cup \cdots \cup S_{r_q}^n(\alpha_q)$$

and construct the set $B \subseteq B^{n+1}$

$$B = S_{r_1}^{n+1}(\beta_1) \cup \cdots \cup S_{r_q}^{n+1}(\beta_q),$$

where $\rho(\alpha_i, \alpha_j) = \rho(\beta_i, \beta_j)$ for $1 \le i, j \le q$. We denote this map $2^{B^n} \mapsto 2^{B^{n+1}}$ by φ. The sets A and $\varphi(A)$ have a lot in common. It is useful to compare the similarity between them with the similarity between $S_k^n(0, \ldots, 0)$ and $S_k^{n+1}(0, \ldots, 0)$.

Theorem 5.2 [16]. *For any critical subset $A \subseteq B^n$ (not necessary optimal) there exists a number $w(A)$ such that for any $t \ge w(A)$ the set $\varphi^t(A) \subseteq B^{n+t}$ is critical and optimal.*

For example the set $S_1^4(0,0,0,0) \cup S_1^4(1,1,1,1)$ is not optimal (compare it with L_{10}^4), but the set $S_1^5(0,0,0,0,0) \cup S_1^5(1,1,1,1,0)$ is optimal, so the number w for it equals 1.

The last theorem in fact states that any construction in a sense may be an optimal subset when the dimension of the space increases. For the number $w(A)$ only rough upper bounds are known.

Now we are ready to present the classification of optimal normalized subsets. Let $A \subseteq B^n$ be such a subset.

Definition 5.2 *We say that A is:*

- *a set of the first type if there is no $B \subseteq B^{n-1}$ such that $A = \varphi(B)$;*
- *a set of the second type if $A = \varphi(B)$ for some optimal set $B \subseteq B^{n-1}$;*
- *a set of the third type if $A = \varphi(B)$ for some nonoptimal set $B \subseteq B^{n-1}$.*

Later we will show the correctness of this definition.

It is clear that a set of the second type exists iff there exists an integer solution x of $x + p(x, n-1) = m$. Now specification of all optimal subsets of this type is reduced to such specification for B^{n-1}. Below we will prove that a subset of the third type may appear only due to the φ transformation. As to subsets of the first type, nothing essential is known. So it is an open problem to derive at least some properties of these subsets.

Now we present some properties of subsets of the third type and prove the correctness of Definition 5.2.

Definition 5.3 *A number m is called the precritical cardinality if m is not a critical cardinality, but $m + 1$ is a critical one.*

By $\alpha(L_m^n)$ we mean, as above, the greatest vector of L_m^n in order $\mathcal{L}$.

Lemma 5.1 *Let $\alpha(L_m^n)$ be of the form $\alpha(L_m^n) = (\alpha_1, \ldots, \alpha_{n-t-1}, 0, \underbrace{1, \ldots, 1}_{t})$. Then*

(i) $p(m, n)$ is a critical cardinality iff $t \ge 2$;
(ii) $p(m, n) - p(m-1, n) = t$.

Proof.
(i) follows from Lemma 4.2 and $\alpha(P(L^n_m)) = (\alpha_1, \ldots, \alpha_{n-t-1}, 0, 0, \underbrace{1, \ldots, 1}_{t-1})$.

(ii) One has

$$\begin{aligned} \alpha(L^n_{m-1}) &= (\alpha_1, \ldots, \alpha_{n-t-1}, 1, 0, \underbrace{1, \ldots, 1}_{t-1}) \quad \text{and} \\ \alpha\left(P(L^n_{m-1})\right) &= (\alpha_1, \ldots, \alpha_{n-t-1}, 1, 0, 0, \underbrace{1, \ldots, 1}_{t-2}). \end{aligned}$$

To complete the proof now, notice that there are exactly $t-1$ vectors in order $\mathcal{L}$ between $\alpha(P(L^n_m))$ and $\alpha(P(L^n_{m-1}))$. ∎

Lemma 5.2 *If m is not a critical cardinality, then $p(m+1,n) - p(m,n) \leq 1$. Moreover, the equality holds iff m is a precritical cardinality.*

Proof.
If $m+1$ is not a critical cardinality then $p(m+1,n) = p(m,n)$ and the Lemma is true. So let $m+1$ be a critical cardinality and assume $p(m+1,n) - p(m,n) > 1$. Then the n-th and $(n-1)$-st entries of $\alpha\left(L^n_{m+1})\right)$ equal 1 by Lemma 5.1. Hence the n-th entry of $\alpha\left(L^n_m\right))$ equals 1 also, i.e. m is a critical cardinality, which is a contradiction. Consequently $p(m+1,n) - p(m,n) \leq 1$.

On the other hand let $p(m,n) + 1 = p(m+1,n)$. Then $m+1$ is a critical cardinality. Let us show that m is not a critical cardinality. Indeed, in the contrary case

$$\alpha\left(L^n_{m+1}\right) = (\alpha_1, \ldots, \alpha_{n-t-1}, 0, \underbrace{1, \ldots, 1}_{t}),$$

where $t \geq 2$. Then $p(m+1,n) - p(m,n) \geq t \geq 2$ by Lemma 5.1. The obtained contradiction completes the proof. ∎

Lemma 5.3 *Let A be an optimal subset and $P\left(A^0(n)\right) \supseteq \pi_n\left(A^1(n)\right)$. Then $C_n(A) = L^n_{|A|}$.*

Proof.
Denote $B = C_n(A)$, $m = |A|$ and $D = L^n_m$. Then $P\left(B^0(n)\right) \supseteq \pi_n\left(B^1(n)\right)$ and B is an optimal set (Lemma 2.1). Since $P\left(B^0(n)\right) \supseteq \pi_n\left(B^1(n)\right)$, then $|D^1(n)| = |B^1(n)|$, hence $|D^0(n)| = |B^0(n)|$. Since $D^\sigma(n)$ and $B^\sigma(n)$ are standard subsets for $\sigma = 0, 1$ (in $n-1$ dimensions), one gets $B = L^n_m$. ∎

Theorem 5.3 *Let $A \subseteq B^n$ be an optimal critical subset of a noncritical cardinality m and there exists an index i for which $p(m_0, n-1) > m_1$ holds. Then*
(i) m is a precritical cardinality and m_0 is a critical one;
(ii) $A^0(i)$ is nonoptimal critical subset for $n \geq 5$;
(iii) $P\left(A^0(i)\right) = \pi_i\left(A^1(i)\right)$.

Proof.
We assume without loss of generality that A is a normalized subset and $i = n$.

(i) Denote $B = C_n(A)$. Then B is an optimal subset and $P\left(B^0(n)\right) \supseteq \pi_n\left(B^1(n)\right)$ (Lemma 3.1). So, B is a standard set by Lemma 5.3. Since m is noncritical cardinality, then $\alpha(B) \in B^0(n)$ (Lemma 4.2). Consider the subset $C = B \setminus \alpha(B)$.

If $|P\left(C^0(n)\right)| < |C^1(n)| = |B^1(n)|$ then there exists a point $\beta \in B^1(n)$, obtained from $\alpha\left(B^0(n)\right)$ by replacing some it's nonzero entry to 0, and the n-th to 1. Hence β lexicographically precedes $\alpha\left(B^0(n)\right)$ and $\|\beta\| = \|\alpha\left(B^0(n)\right)\|$, and so $\alpha\left(B^0(n)\right) \neq \alpha(B)$, which is a contradiction.

Therefore $|P\left(C^0(n)\right)| \geq |C^1(n)|$. Assume the strict inequality holds here, i.e. there exists a point $\beta \in P\left(C^0(n)\right)$ such that $\pi_n(\beta) \notin C^1(n)$. Then consider the set D, obtained from B by replacing $\alpha\left(B^0(n)\right)$ to $\pi_n(\beta)$. One has $|P(D)| > |P(B)|$, and B is nonoptimal set.

Consequently, $|P\left(C^0(n)\right)| = |C^1(n)|$ and so m_0 is a critical cardinality. If now the n-th entry of $\alpha(L^n_{m+1})$ equals 0 (i.e. $m+1$ is a noncritical cardinality), then consider a point $\beta \in P\left(B^0(n)\right) \setminus P(B)$ (such point exists since $p(m_0, n-1) > m_1$) and the set $E = B \cup \pi_n(\beta)$. One has $|P(E)| > |P(L^n_{m+1}|)$, which is a contradiction. Therefore $m+1$ is a critical cardinality, and so m is precritical one.

(ii) We use induction on n and let $i = n$ again. Moreover, let

$$|A^0(n)| = \sum_{t=0}^{k} \binom{n-1}{t} + \delta_0, \quad |A^1(n)| = \sum_{t=0}^{k-1} \binom{n-1}{t} + \delta_1.$$

The proof of (ii) for $n = 5$ is left to the reader. In this case either $m = 12$ or $m = 14$. If $m = 12$, then A is the union of two balls of radius 1, for which (ii) holds. If $m = 14$, then there are only two nonisomorphic sets:

$$\begin{aligned} A_1 &= \{0, 1, 2, 4, 8, 16, 17, 18, 20, 24, 3, 7, 11, 19\} \quad \text{and} \\ A_2 &= \{0, 1, 2, 4, 8, 16, 24, 25, 26, 28, 3, 7, 11, 19\}, \end{aligned}$$

where the vertices of A_1, A_2 are represented by their lexicographic numbers.

Let us make the inductive step for $n \geq 6$. Assume now that $A^0(n)$ is a normalized set (in $n-1$ dimensions). Therefore we may apply Theorem 4.4 to $A^0(n)$. By this theorem there exists an index j for which

$$\begin{aligned} |A^{00}(n,j)| &= \sum_{t=0}^{k} \binom{n-2}{t} + \left(\delta_0 \triangle \binom{n-2}{k}\right), \\ |A^{01}(n,j)| &= \sum_{t=0}^{k} \binom{n-2}{t} - \left(\binom{n-2}{k} \triangle \delta_0\right). \end{aligned}$$

Case 1. Let $\delta_0 \leq \binom{n-2}{k}$. Then $A^{00}(n,j) = S^{n-2}_k(0,\ldots,0)$ and since $S^n_k(0,\ldots,0) \subseteq A$ (Theorem 3.3) and $P\left(A^0(n)\right) \supseteq \pi_n\left(A^1(n)\right)$, one has $A^{10}(n,j) = S^{n-2}_{k-1}(0,\ldots,0)$, i.e. $A^0(j) = S^{n-1}_k(0,\ldots,0)$. Moreover, $p(|A^{00}(n,j)|, n-2) \leq |A^{01}(n,j)|$ and $\delta_1 \leq \binom{n-2}{k-1}$ imply $p(|A^0(j)|, n-1) \leq |A^1(j)|$. Therefore for the set A and j the statement (i) of Theorem 3.1 holds, and so $A^1(j)$ is an optimal subset. Since $A^{01}(n,j)$ is a critical subset and $A^{01}(n,j) \supseteq \pi_n\left(A^{11}(n,j)\right)$, then $A^1(j)$ is a critical subset. It is easy to show that $|A^1(j)|$ is noncritical cardinality. Finally, since

$$p(|A^{00}(n,j)|, n-2) = |A^{10}(n,j)|\,,$$

then

$$p(|A^{01}(n,j)|, n-2) > |A^{11}(n,j)|\,.$$

Consequently for the subset $A^1(j)$ and n the statement (ii) of Theorem 5.3 holds. Then $A^{01}(n,j)$ is nonoptimal subset by the induction, which is a contradiction

Case 2. Let now $\delta_0 \geq \binom{n-2}{k}$. Then $A^{01}(n,j) = S_k^{n-2}(\gamma)$ for some γ, $\|\gamma\| \leq 1$ (here the norm is computed in the $(n-2)$-subcube). Since

$$|A^{01}(n,j)| \geq |P\left(A^{00}(n,j)\right)| = p\left(|A^{00}(n,j)|, n-2\right),$$

then $A^0(j)$ is an optimal subset by Theorem 3.1(i). Taking into account $P\left(A^{00}(n,j)\right) \supseteq \pi_n\left(A^{10}(n,j)\right)$ and that $A^{00}(n,j)$ is a critical set, one gets that $A^0(j)$ is an optimal critical subset. Our goal now is to prove that $|A^0(j)|$ is noncritical cardinality in order to apply to it the inductive hypothesis.

For that purpose notice that $A^{11}(n,j) \subseteq S_{k-1}^{n-2}\left(\pi_n(\gamma)\right)$ and let us show that in fact the equality holds here. Assume the strict inclusion holds. Denote $B = C_n(A)$ and consider the set D obtained from A by replacing $A^{\sigma\tau}(n,j)$ $(\sigma, \tau \in \{0,1\})$ to standard sets in $(n-2)$ dimensions. It is not difficult to show that D is an optimal subset. If now $|P\left(B^0(n)\right)| - |B^1(n)| \geq 3$, then $|B^1(n)| = p(m,n)$ is a critical cardinality (Lemma 5.1), and so A is noncritical subset, which is a contradiction. Therefore,

$$1 \leq |P\left(B^0(n)\right)| - |B^1(n)| \leq 2.$$

Assume first that this difference equals 1. Then

$$D^{11}(n,j) = S_{k-2}^{n-2}(0,\ldots,0) \setminus \epsilon$$

for some ϵ. So, for $k \geq 2$ one has $|P(D \cup \epsilon)| - |P(D)| \geq 2$, which is impossible by Lemma 5.2. If $k = 1$, then $A^{11}(n,j) = \emptyset$ and A is not a critical set since $|P\left(A^{00}(n,j)\right)| < |A^{01}(n,j)|$.

Finally, if $|P\left(B^0(n)\right)| - |B^1(n)| = 2$, then it is sufficient to consider the case

$$D^{11}(n,j) = S_{k-2}^{n-2}(0,\ldots,0) \setminus \{\epsilon_1, \epsilon_2\}\,.$$

Then $k \geq 2$ and $|P(D \cup \epsilon_1)| - |P(D)| \geq 2$ as above.

Therefore, we proved $A^{11}(n,j) = S_{k-1}^{n-2}(\pi_n(\gamma))$, and so $A^1(j) = S_k^{n-1}(\gamma)$. Consequently,

$$|A^0(j)| = \sum_{t=0}^{k} \binom{n}{t} + \delta - \sum_{t=0}^{k} \binom{n-1}{t} = \sum_{t=0}^{k} \binom{n-1}{t} + \delta',$$

where $\delta' = \delta - \binom{n-1}{k} \geq 0$, which implies that $|A^0(j)|$ is noncritical cardinality (since m is noncritical).

Hence, we may apply the inductive hypothesis to $A^0(j)$ and x_n and get that $A^{00}(n,j)$ is nonoptimal subset, which is a contradiction.

To complete the proof of (ii) we have to consider the case when $A^0(n)$ is not j-normalized. If now $\delta_0 < \binom{n-2}{k}$, then $|A^{00}(n,j)| < \sum_{t=0}^{k}\binom{n-2}{t}$, which contradicts to $S_k^n(0,\ldots,0) \subseteq A$. If $\delta_0 \geq \binom{n-2}{k}$, then $|A^{11}(n,j)| \geq \sum_{t=0}^{k-1}\binom{n-2}{t}$, and if at least one of the last two inequalities is strict then A is not n-normalized by x_i. If the equality holds in both of them, then $|A|$ ia a critical cardinality.

Consequently in all the cases the assumption that $A^0(n)$ is an optimal subset leads us to a contradiction.

(iii) Assume the contrary, i.e. $P\left(A^0(i)\right) \supset \pi_i\left(A^1(i)\right)$. Hence there exists a point $\alpha \in P\left(A^0(i)\right) \setminus \pi_i\left(A^1(i)\right)$. Denote $\beta = \pi_i(\alpha)$ and $B = A \cup \beta$. By Lemma 5.2, $p(m+1,n) = p(m,n)+1$ and since $|P(B)| \geq |P(A)|+1$, then B is an optimal subset. Moreover, $|B| = m+1$ is a critical cardinality by (i). Therefore, $B^0(i) = A^0(i)$ is an optimal subset (Theorem 3.1c). But this contradicts to (ii) and completes the proof of the Theorem. ∎

Now we are ready to prove the correctness of Definition 5.2. It is sufficient to prove that a set A cannot be of type 2 and 3 simultaneously. Assume in contrary, that there exists coordinates x_i and x_j for which propositions of Theorem 3.1b and c respectively hold. From Theorems 5.2(iii) and 3.1c it follows that $A^1(i) = \pi_i(P(A^0(i))$ and $A^1(j) = \pi_i(P(A^0(j))$. Consequently, $|A^{01}(i,j)| = |A^{10}(i,j)|$, and so $A^0(i)$ and $A^0(j)$ are in fact the same sets in $n-1$ dimensions. But on the one hand one of them is optimal while the other is not, which is a contradiction.

6 Edge Isoperimetric Problem

Up to now we were concerned with the vertex version of the isoperimetric problem, where one has to minimize a number of some vertices. Here we deal with another type of this problem, where one has to minimize some number of edges.

Let A be a set of vertices of B^n.

Definition 6.1 *An edge $e = (\alpha,\beta)$ of B^n is called the boundary edge of A if either $\alpha \in A$, $\beta \notin A$ or $\beta \in A$, $\alpha \notin A$ holds. If the both ends of e are in A, then we call e the inner edge of A.*

Denote by E(A) (I(A)) the collection of all boundary (inner) edges of A. Consider the problem to find an m-element subset $A \subseteq B^n$ with minimal value of $|E(A)|$. We call such a set E-optimal. The following result was proved by different authors [9, 37, 39].

Theorem 6.1 *The collection of the first m vertices in the lexicographic order is E-optimal subset for all m. Moreover, all the other E-optimal subsets are isomorphic to it.*

It is interesting that this problem is much simpler that the vertex one, and all it's solutions are known.

It is also possible to specify all the subsets with maximal value of $|E(A)|$. Namely, denote by η a numeration of vertices of B^n, which first enumerates arbitrarily all the vertices with even number of ones and then the rest.

Theorem 6.2 [37]. *The initial segment of η of length m gets the maximum for $|E(A)|$.*

It is also proved in [37] that there are no other optimal numberings. The two listed problems are also interesting from another point of view. It is clear that $|I(A)|+|E(A)| = n\cdot|A|$ for the n-cube, and so we know how to maximize and minimize $|I(A)|$. The problem for $I(A)$ has applications in coding theory [10]. Moreover it is a partial case of a very general kind of problems of finding the length or width of a graph [37].

Definition 6.2 *A set of vertices $A \subseteq B^n$ is called the ideal if the conditions $\alpha \in A$ and $\beta \prec \alpha$ imply $\beta \in A$.*

It is easy to prove that in order to find an m-element set A with maximal possible value of $|I(A)|$ (we call such a set I-optimal) it is sufficient to consider ideals only. But for an ideal A the number of it's inner edges may be computed as $\sum_{i=0}^{n} i\cdot|A_i|$, where A_i is the collection of vertices of A exactly with i ones.

So, we came to very important problem to find an m-element ideal with maximal value of $F = \sum_{i=0}^{n} w_i\cdot|A_i|$, where w_i are given nonnegative numbers. It is not a pure edge isoperimetric problem, but of the same type. This problem for B^n and different sequences $\{w_i\}$ was solved in [1]. The extension of it for the case of the k-valued cube one can find in [21], where it is a consequence of a more general problem. If $\{w_i\}$ is a monotone nondecreasing sequence, then the corresponding problem of maximizing of F was solved in [31, 44, 53] for the case of the of k-valued cube, and in [52] for the case of the poset of faces of the n-cube ordered by inclusion. Finally, in [18] it is proved that this problem is immediate consequence of the Kruskal-Katona theorem, which is proved there for some wider class of posets. An interesting application of the considered problem to maximizing of the value of determinants one can find in [32].

Let us return back to the original problem of maximizing of $I(A)$. We say, that an edge $e = (\alpha,\beta)$ is of i-th direction if α and β differ in the i-th entry. Let $D = \{d_{i_1},\dots,d_{i_t}\}$ be a set of directions. Consider now a problem of finding an m-element subset $A \subseteq B^n$ with maximal value of $|I_D(A)|$, where $I_D(A)$ consists of those edges in $I(A)$, whose direction belongs to D. In order to construct a solution now, decompose B^n into t-dimensional subcubes, direction of edges of which belong to the set $\{1,\dots,n\}\setminus D$. Denote these subcubes by $B_1,\dots,B_{2^{n-t}}$. Represent the number m in the form $m = p\cdot 2^t + q,\ 0 \le q < 2^t$ and choose $p+1$ subcubes $B_{i_1},\dots,B_{i_{p+1}}$. Consider now set A_D, which is the union of all vertices of some p subcubes and the collection of q first vertices in the lexicographic order in the $(p+1)$-st one. Then the maximum of $|I_D(A)|$ is achieved on A_D [17].

Of course, one may consider the edges of the unit cube as faces of dimension 1 and state an analogous question of constructing a subset of vertices of B^n of a given cardinality m, containing maximal possible number of faces of dimension d for $d > 1$. Here by a face we mean a subset of B^n determined by fixing the values of some coordinates and allowing the reminder free rein. It follows, for example, from [21, 23, 24] that the collection of the first m vertices in the lexicographic order is the best choice. In fact this problem is a partial case of the Kruskal-Katona problem (see Section 7d) for some special poset. Refer to [18] for more details.

Finally let us mention that an edge of B^n may be considered as a pair of points on distance 1. That allows us to state the Edge Isoperimetric Problem, for example, for the

k-valued n-dimensional cube. In [51] one can find analogs of Theorems 6.1 and 6.2 for this case.

7 Related Results and Applications

In this Section we consider several isoperimetric type problems, list known results about their solutions and show usefulness of the isoperimetric approach for some other discrete extremal problems.

a. Different Kinds of Boundary Operators.

Up to now we dealt with the Γ operator. The following three are also very useful. For $A \subseteq B^n$ and $p \geq 1$ denote

$$\begin{aligned} \Gamma_p(A) &= \left\{\alpha \in A : S_p^n(A) \not\subseteq A\right\}, \\ G_p(A) &= \{\alpha \in B^n \setminus A : \rho(\alpha, A) \leq p\}, \\ \mathrm{Bdry}(A) &= \{\alpha \in B^n \setminus A : \alpha - e_i \in A \ \text{ for some } i,\ 1 \leq i \leq n\}, \end{aligned}$$

where e_i is the vector of B^n with the only one nonzero entry, namely the i-th. When $p = 1$ we will omit the subscript of Γ and G operators for brevity. Consider the problems of finding an m-element set $A \subseteq B^n$ with minimal values of $|\Gamma_p(A)|,\ |G_p(A)|$ and $|\mathrm{Bdry}(A)|$. Denote these problems by $I_{\Gamma_p},\ I_{G_p}$ and I_{Bdry} respectively.

Lemma 7.1 [54]. *For any $A \subseteq B^n$*

$$G_p(A) = \Gamma_p(B^n \setminus A).$$

Therefore each solution of I_{G_p} corresponds to some solution of I_{Γ_p} and vice versa. From Theorem 3.4 it follows

Theorem 7.1 [16]. *If $A \subseteq B^n$ is a solution of I_{G_p}, then it is also a solution of I_{G_q} for $q \geq p$.*

It is interesting that $B^n \setminus L_m^n \equiv L_{2^n-m}^n$, hence L_m^n is a solution of I_{G_p} as well as of I_{Γ_p} for any p.

Theorem 7.2 [60]. *L_m^n is a solution of I_{Bdry}.*

Therefore our L_m^n is three times optimal set.

In order to determine whether α is an inner point of A for I_{Γ_p} and I_{G_p} we used balls. The natural question is why balls ? Imagine that for each point $\alpha \in B^n$ one has a subset $U(\alpha) \subseteq B^n$ and the point α is inner point of a set A iff $U(\alpha) \subseteq A$. For I_{Γ_p} we have $U(\alpha) = S_p^n(\alpha)$. Consider now the problem I_Γ with $U(0,\ldots,0)$ as a proper subset of $S_1^n(0,\ldots,0)$ and let $U(\alpha) = U(0,\ldots,0) \oplus \alpha$, where $\oplus$ means componentwise modulo 2 addition. Denote this isoperimetric problem by I_Γ^*. A solution of I_Γ^* one can find in [14].

An analogous construction already appeared in the last paragraph of Section 6. The set $U(0,\dots,0)\setminus(0,\dots,0)$ may be considered as an analog of the directions set D from there. So, represent the number m in the form $m = p\cdot 2^t + q$, $0 \le q < 2^t$ again and choose $p+1$ subcubes $B_{i_1},\dots,B_{i_{p+1}}$, defined as in the Section 6. Consider now set A, which is the union of all vertices of some p subcubes and the standard set L_q^t in the $(p+1)$-st one. Then this set A is a solution of I_Γ^*.

It easily follows from [14] how to solve a slightly more general problem for the case when $U(0,\dots,0)$ is an arbitrary collection of linear independent vectors of B^n (see also [17]).

b. Isoperimetric Problems for Different Structures.

Each of the following problems deserves a separate consideration and analysis, so we will be restricted by brief references only. The proof techniques for these cases is similar in general to the n-cube's one, but nevertheless some features are very different.

The most analogous structure to B^n is the n-dimensional rectangle which may be defined as the Cartesian product of n paths of different lengths (finite or infinite). A solution of the isoperimetric problem on it with Bdry operator is in [59]. In [60] an analogous problem with the same boundary operator was solved for the torus (Cartesian product of n circles). In these two papers analogs of L_m^n were introduced and optimality of them was proved. In [19] it was shown that the same set is a solution of isoperimetric problem with G operator. In [27] the authors proved good isoperimetric inequalities, from which it follows that a ball centered in the zero point gets the minimum for the Γ operator.

In [40] It was solved the isoperimetric problem for the torus and Γ operator, when all the circles in the Cartesian product are of even length or infinite. The solution is described also in the form of a linear order. Here we present the definition of this order. Given integers $k_1,\dots,k_n$, $(1 \le k_1 \le k_2 \dots \le k_n \le \infty)$, denote

$$T^n = \{(x_1,\dots,x_n) : -k_i+1 \le x_i \le k_i,\ 1 \le i \le n\}.$$

For $x \in T^n$ denote $|x| = (|x_1|,\dots,|x_n|)$, $\sigma(x) = (\sigma_1,\dots,\sigma_n)$, where $\sigma_i = 1$ if $x_{n-i+1} > 0$ and 0 otherwise and let $N(x) = \sum_{i=1}^n |x_i|$. Consider the order $\mathcal{T}$ on the set T^n defined as follows. We say x precedes y in order $\mathcal{T}$ iff

1. $N(x) < N(y)$, or
2. $N(x) = N(y)$ and $\sigma(y)$ precedes $\sigma(x)$ lexicographically, or
3. $N(x) = N(y)$, $\sigma(x) = \sigma(y)$ and $|y|$ precedes $|x|$ lexicographically.

In [40] it was proved that any initial segment of this order is a solution of the corresponding isoperimetric problem. Later the same author showed that omitting the restrictions on the length of the circles leads to absence of the linear order. In [29] one can find an isoperimetric inequality for this case.

Let us consider now more strange structure, namely the collection of all vertices of the unit cube with even number of ones, i.e.

$$B^{n,0} = \{\alpha = (\alpha_1,\dots,\alpha_n) \in B^n : \|\alpha\| \equiv 0 \pmod 2\}.$$

Given am integer m, $1 \le m \le 2^{n-1}$, find an m-element subset $A \subseteq B^{n,0}$ with minimal value of $|G(A)|$ among all the subsets of the same cardinality. This problem was solved by different authors [14, 28, 48, 49, 57]. In fact this problem is equivalent to our original problem and here we show the equivalence.

Let $A^0(i)$ and $A^1(i)$ be subsections of A. Notice that $A^0(i)$ is a subset of even levels of the corresponding $(n-1)$-subcube and $A^1(i)$ of odd ones. So, denoting $A^* = A^0(i) \cup \pi_i\left(A^1(i)\right)$, one has $|A| = |A^*|$.

Lemma 7.2 [14, 57]. $|G(A)| = |G(A^*)| + |A|$, *where the G operator in the right hand side works in $(n-1)$ dimensions.*

Therefore we may apply to $A^0(i)$ the Main Isoperimetric Theorem (in $(n-1)$ dimensions and with G operator) and then construct the n-dimensional set by shifting all the vertices with odd number of ones to the corresponding $(n-1)$-subcube without increasing of G and obtain in such a way the "even" analog of L_m^n.

Theorem 7.3 *Denote by $L_m^{n,0}$ the m-element subset of $B^{n,0}$, which is intersection of a standard set with $B^{n,0}$. Then $|G(L_m^{n,0})| \leq |G(A)|$ for any $A \subseteq B^{n,0}$, $|A| = m$.*

Of course, the same holds for the "odd" analog of L_m^n.

This theorem was used in [45] to find the asymptotic formula for the number of binary codes with distance 2. In [48] it was considered a very similar isoperimetric problem on a set of codewords with distance 2 instead of "even" vertices. As it was shown there for the minimization of G in this case it is sufficient to consider the "even" vertices only. Besides it was mentioned an interesting application of this theorem to the theory of multiuser communication. The proof techniques of [48] differs of [14, 57] and is more complicated. Later the authors of [48] found much simpler proof of Theorem 7.3 [49], which is similar to [14] and [57]. In [28] one can find corresponding isoperimetric inequality.

The next structure we consider is the sphere of radius t of Hamming space centered in the zero point. We call this set also the t-th level of B^n and denote it by B_t^n. The problem is to find an m-element subset $A \subseteq B_t^n$ with minimal value of $|G(A)|$. It is clear that for $t = 1$ each subset is optimal. For $t = 2$ in [11] it was proved that the collection of the last m points in the LEX order (i.e. the final LEX segment) is a solution. Ibid it was proposed a set suspicious to optimality for lager t. But as it turned out it is not optimal for $t \geq 3$, and even is not optimal in asymptotical sense [12]. As to asymptotical investigations, since

$$|G(A) \cup B_{t-1}^n| = o(|G(A) \cup B_{t+1}^n| \quad \text{when } t = o(\sqrt{n})$$

as $n \to \infty$, we may pay attention to minimization of the "upper part" of G only. Due to Kruskal-Katona theorem (see subsection **e** below) it is clear that the final LEX segment is asymptotically optimal set when $t = o(\sqrt{n})$. In [12] it was proved that it is not asymptotically optimal when $t = O(n)$.

c. Asymptotic Estimations.

Denote $\mathcal{M}_a = \{A \subseteq B^n : |A| = a,\ 0 \leq a = 2^{n-1}(1 + \alpha(n)) \leq 2^n\}$ and $\mathcal{M} = \cup \mathcal{M}_a$. We call a set $A \subseteq B^n$ γ-dense if $\gamma(A) = |P(A)|/|A| = \gamma, \quad 0 \leq \gamma \leq 1$. Further denote by $m(\gamma) \quad (m_a(\gamma))$ the fraction of those subsets $A \in \mathcal{M} \quad (A \in \mathcal{M}_a)$, which are γ-dense. Denote respectively by $\tilde{m}(\gamma)$ and $\tilde{m}_a(\gamma)$ the fraction of those $A \in \mathcal{M}$ and $A \in \mathcal{M}_a$, for which $\gamma(a) \geq \gamma$. The following theorem specifies typical values of $|P(A)|$.

Theorem 7.4 [7,8].

a. $\lim_{n\to\infty}\gamma(A)=0$ *for almost all subsets* $A\subseteq B^n$. *If* k *is arbitrary integer, then*

$$\lim_{n\to\infty} m\left(\frac{k}{2^{n-1}}\right)=\frac{1}{2^k\,k!\sqrt{e}} \quad \textit{and} \quad \lim_{\eta(n)\to\infty}\tilde{m}\left(\frac{\eta(n)}{2^n}\right)=0;$$

b. If $\lim_{n\to\infty} n\alpha(n)=-\infty$, *then* $\gamma(A)=0$ *for almost all* $A\in\mathcal{M}_a$*;*

c. If $\lim_{n\to\infty} n\alpha(n)=-\infty$, *then* $\lim_{n\to\infty} m_a\left(\frac{1+\alpha(n)}{2}\right)^n=1$. *If* $1-\alpha(n)\sim\frac{c}{n}$, $c>0$, *then* $\gamma(A)\sim e^{-c/2}$ *for almost all* $A\in\mathcal{M}_a$. *Moreover,* $m_a(1)\to 1$ *if* $1-\alpha(n)=o\,(1/n)$*;*

d. If $\lim_{n\to\infty} n\alpha(n)=\infty$, *then for almost all* $B\in\mathcal{M}_b$, $|b-a|=o\,(2^n/n)$ *one has* $\gamma(B)\sim\left(\frac{1+\alpha(n)}{2}\right)^n\sim\gamma(A)$*;*

e. If $\lim_{n\to\infty} n\alpha(n)=\lambda$, *then* $m_a\left(\frac{k}{2^{n-1}}\right)\to\frac{1}{k!}\left(\frac{e^\lambda}{2}\right)^k e^{-\frac{e^\lambda}{2}}$, $\quad\tilde{m}_a\left(\frac{\eta(n)}{2^{n-1}}\right)\to 0$ *as* $\eta(n)\to\infty$.

The mathematical expectation and the variance of the number of inner points of subsets from $\mathcal{M}_a$ equal [7]

$$\begin{aligned}\mathbf{Ex}_a &= 2^n\binom{2^n-n-1}{a-n-1}\Big/\binom{2^n}{a} \quad \text{and}\\ \mathbf{D}_a &= 2^n\,\frac{\binom{2^n-n-1}{a-n-1}}{\binom{2^n}{a}}+2^n\left(n+\binom{n}{2}\right)\frac{\binom{2^n-2n}{a-2n}}{\binom{2^n}{a}}+\end{aligned}$$

$$2^n\left(2^n-1-n-\binom{n}{2}\right)\frac{\binom{2^n-2n-2}{a-2n-2}}{\binom{2^n}{a}}-\left(2^n\,\frac{\binom{2^n-n-1}{a-n-1}}{\binom{2^n}{a}}\right)^2.$$

Up to now we were concerned with minimization of the boundary. If one wish to maximize it, it leads to problems of coding theory. But if we restrict the structure of subsets by which the boundary is maximized, then there arise some related problems. In [46] the maximization boundary problem was investigated for ideals in the cube.

Theorem 7.5 [46]. *Let* I *denotes the collection of all ideals of* B^n. *Then*

$$0{,}011\cdot 2^n\,\frac{\ln^{3/2} n}{\sqrt{n}}\le\min_{A\in I}(2^n-|G(A)|)\le 1{,}5\cdot 2^n\,\frac{\ln^{3/2} n}{\sqrt{n}}\,.$$

This theorem has an application to the relative database systems [46].

In [47] it was obtained an upper bound for the maximal boundary of an antichain in B^n, and in [30] one can find a lower bound. We summarize the both results in the following

Theorem 7.6 [30,47]. *Let* U *denotes the collection of all antichains of* B^n. *Then*

$$0{,}2\cdot 2^n\le\max_{A\in U}|G(A)|\le 0{,}9987\cdot 2^n\,.$$

Finally in [56] one can find a lot of asymptotic results about the number of subsets of B^n with given cardinality of the boundary. Some results of such a type were applied later for determination the number of monotone functions in different posets.

d. The Kruskal-Katona Theorem.

Consider again the collection B_i^n of all vectors of B^n with exactly i ones, and let $A \subseteq B_i^n, \quad |A| = m$. Our goal is to find am m-element subset of B_i^n with minimal possible value of $|T(A)|$, where $T(A) = G(A) \cap B_{i+1}^n$. We call such a set A i-optimal. By the Kruskal-Katona theorem [41,50] the collection of m last vectors of B_i^n in the lexicographical order is an i-optimal subset. This result, however, immediately follows from the Main Isoperimetric Theorem. Indeed, denoting $M(m) = \sum_{t=0}^{i-1} \binom{n}{t} + m$, one has

$$|G(L_{M(m)}^n)| = \binom{n}{i} - m + |T(L_{M(m)}^n \cap B_i^n)| \,,$$

and the reduction follows. Moreover, the known information about optimal subsets can be applied to i-optimal one. Namely we can determine some cases of the uniqueness of the i-optimal subset (up to isomorphism).

Theorem 7.7 *If $M(m)$ is a critical cardinality, then there exists the only one i-optimal m-element subset of B_i^n. If $p(M(m), n)$ is a critical cardinality, then the family of all m-optimal sets may be obtained by adding to $L_{M(m)}^n$ arbitrary $M(m) - |S(L_{M(m)}^n|$ points of B_i^n.*

The first proposition of this Theorem follows from Theorem 4.3 and was appeared in [36] too. The second one easily follows from Theorem 4.7.

We can also consider the problem of minimization of $T_l(A) = T(\cdots T(A) \cdots)$ (l times). The following theorem was also appeared in [36].

Theorem 7.8 [16]. *If $A \subseteq B_i^n$ gets the minimum for $T_l(A)$, then it also gets the minimum for $T_s(A)$ for $s \geq l$.*

Therefore, the Kruskal-Katona theorem is very closed to the isoperimetric one. It has a lot of applications and was proved for a number of posets. Here we only mentioned this powerful theorem. Much more information one can find in the forthcoming paper [18].

e. Three Problems from Computer Science.

The following extremal problem was considered in [2] and is concerned with unreliable networks in parallel computers. Denote by $\lambda(c) \quad (\mu(c))$ the maximal number with the property that removal of any $m \leq \lambda(c)$ vertices ($m \leq \mu(c)$ edges) from B^n results in a graph with a maximal component containing at least c vertices (edges). The problem is to derive estimations of $\lambda(c)$ and $\mu(c)$.

Let us consider here the vertex version of the problem. Denoting $\lambda^*(c) = \lambda(c+1)+1$, suppose that the removal of $\lambda^*(c)$ vertices results in the connected components $Z_1, \ldots, Z_l$. One has

$$\lambda^*(c) \geq |\bigcup_{i=1}^{l} (G(Z_i) \setminus Z_i)| \,.$$

Thus we may use the isoperimetric results to estimate $\lambda^*(c)$. Denote $N_k = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{k+1}$.

Theorem 7.9 [2].

a. $\lambda^*(N_k) = \binom{n}{k}$ *for* $N_k \geq \frac{2}{3}2^n$;

b. $\binom{n}{k-1} \leq \lambda^*(c) \leq \binom{n}{k}$ *if* $\max(\frac{2}{3}2^n, N_k) < c \leq N_{k-1}$;

c. Define $k_0 = \max\{k : 2^n - N_k < \frac{1}{3}2^n\}$ *and* $L_0 = 2^n - N_{k_0}$. *Then for any c,* $2^n - 2L_0 \leq c \leq 2^n - L_0$

$$\binom{n}{k_0} \leq \lambda^*(c) \leq 2\binom{n}{k_0}.$$

The analogous propositions can be obtained for the edge version of this problem [2]. One should only use the Edge Isoperimetric Theorem.

The second problem is also about removal vertices from B^n. But now the claim is that no connected component has a pair of complementary vertices (i.e. the Hamming distance between which equals n). What is the least number $K(n)$ of vertices one has to remove ?

Theorem 7.10 [42]. $K(n) = \binom{n}{[n/2]}$.

Here we demonstrate how the isoperimetric approach allows to solve this problem for the two component case [42]. The solution is based on the three following arguments:

1. If the Theorem did not hold, of necessity one of the two components would have size at least $\frac{1}{2} \cdot (2^n - \binom{n}{[n/2]} + 1)$;

2. By the Main Isoperimetric Theorem, every set A whose size lies between this lower bound and 2^{n-1} has $|G(A)|$ at least $\binom{n}{[n/2]}$;

3. The removal vertices must, of course, include the boundaries of all connected components of the remaining vertices.

The solution in the general case claims some additional arguments, but is based also in the isoperimetric approach [42].

The third problem belongs to the area of scheduling theory [20]. Assume that we numerate the edges of B^n by numbers $1, \ldots, n2^{n-1}$. Then for a vertex $v \in B^n$ denote $f(v) = \max_{e_1,e_2} |e_1 - e_2|$, where e_1, e_2 are the labels of edges incident with v and the maximum is taken over all edges of B^n incident with v. The problem is to create a numeration φ with minimal value of $S_\varphi = \sum_{v \in B^n} f(v)$ or $W_\varphi = \max_{v \in B^n} f(v)$. The isoperimetric approach helps to estimate S_φ and W_φ.

If A is a subset of edges of B^n, denote by $V(A)$ the set of vertices incident with A. Further, denote $g(t) = \min_{|A|=t} |G(V(A))|$. Then $S_\varphi \geq \sum_{t=1}^{n2^{n-1}} g(t)$, which helps to derive a low bound for S_φ. Notice that the computation of $g(t)$ is an isoperimetric type problem. An analogous techniques was used for deriving bounds on W_φ. Moreover, the information about all the optimal subsets for some cardinalities allows to find exact values of S_φ and W_φ for small n.

f. Maximally Distant Subsets.

In this Section we consider a problem, which is in fact equivalent to the isoperimetric one. Assume we have to find two subsets $A, B \subseteq B^n$, $|A| = p$, $|B| = q$ which get the maximum to $\rho(A, B) = \min\limits_{\substack{\alpha \in A \\ \beta \in B}} \rho(\alpha, \beta)$. Denote by F_m^n the collection of the last m points in order $\mathcal{L}$.

Theorem 7.11 [6,35]. $\rho(L_p^n, F_q^n) \geq \rho(A, B)$ *for any* $A, B \subseteq B^n$, $|A| = p$, $|B| = q$.

The following argument lies in [6] to derive this theorem from the Main Isoperimetric Theorem. It is clear that if $B = B^n \setminus A$, then $\rho(A, B) = 1$. Moreover, if $P(B) \neq \emptyset$, then $\rho(A, B \setminus \Gamma(B)) = \rho(A, B) + 1$. So, in order to maximize $|B|$ one should minimize $|\Gamma(B)|$.

In [35] the authors first found the two extremal subsets A, B, and after that showed the implication of the Main Isoperimetric Theorem. That is why the two problems are equivalent. The proof in [35] may be considered as one more short proof of the Main Isoperimetric Theorem. It also uses shift techniques.

Finally let us mention that the two sets distance problem was solved in [54] for the case when both p and q are spherical cardinalities and was used later in [55] in the variation principle of Boolean algebra.

g. Maximally Compressed Sets.

The last application we would like to mention here is the problem to find all the subsets of B^n with fixed diameter d, (i.e. $\max\limits_{\alpha,\beta \in A} \rho(\alpha, \beta)$), which have maximal possible cardinality. We call such subsets d-maximal.

One of the d-maximal subsets was found for example in [1]. It is proved there, that if $d = 2t$ then a ball $S_t^n(\alpha)$ is d-maximal subset. If $d = 2t + 1$, then $S_t^n(\alpha) \cup S_t^n(\beta)$ is d-maximal subset for any α, β, $\rho(\alpha, \beta) = 1$. But a success in isoperimetric problems allowed to prove that there are no other d-maximal subsets if $d < n - 1$ [15]. To show it, one have to notice that if $d(A) = d$, then $A \cap (\bar{A} \cup G_{n-d-1}(\bar{A})) = \emptyset$. Here $\bar{A}$ denotes the subset obtained from A by replacing each it's vertex to the opposite one. The last equality implies $2|A| + |G_{n-d-1}(A)| \leq 2^n$, from which it is more or less clear that in order to maximize $|A|$ one should minimize $|G_{n-d-1}(A)|$, and so A is a solution of $I_{G_{n-d-1}}$ (see subsection **a**). But in fact A in our case is a solution of I_Γ [15], and we know it's cardinality, which is a critical one. This helps us to find all the d-maximal subsets.

References

[1] Ahlswede R., Katona G.O.H. *Contributions to the Geometry of Hamming Spaces*, Discr. Math., **17**, No.1(1977), 1–22.

[2] Ahlswede R., Koschnick K.U. *Note on an Extremal Problem Arising for Unreliable Networks in Parallel Computers*, Discr. Math., **47**(1983), 137–152.

[3] Aslanjan L.A., Karachanjan V.M., Torosjan B.E. *A Solution of the Discrete Isoperimetric Problem* (in Russian), Doklady AN Arm. SSR, **LXV**, No.5(1977), 257–262.

[4] Aslanjan L.A., Karachanjan V.M. *A General Specification of Solutions of the Discrete Isoperimetric Problem* (in Russian), Doklady AN Arm. SSR, **LXV**, No.4(1977), 211–215.

[5] Aslanjan L.A., Karachanjan V.M., Torosjan B.E. *On the Compactness of Subsets of Vertices of the n-dimensional Unit Cube*, Soviet Math. Dokl., **19**, No.4(1979), 781–785.

[6] Aslanjan L.A. *An Isoperimetric Problem and Related Extremal Problems for Discrete Spaces* (in Russian), in Problemy Kibernetiki 36, Nauka, Moscow 1979, 85–127.

[7] Aslanjan L.A. *The Discrete Isoperimetric Problem - Asymptotic Case* (in Russian), Doklady AN Arm. SSR, **LXXIV**, No.3(1982), 99–103.

[8] Aslanjan L.A., Akopova I.A. *On the Distribution of the Number of Interior Points in Subsets of the n-Dimensional Unit Cube*, in 37. Finite and Infinite Sets, Eger (Hungary) (1981), 47–58.

[9] Bernstein A.J. *Maximally Connected Arrays on the n-Cube*, SIAM J. Appl. Math., **15**, No.6(1967), 1485–1489.

[10] Bernstein A.I., Steiglitz K., Hopcroft J.E. *Encoding of Analog Signals for Binary Symmetric Channels*, IEEE Trans. Info. Theory, **11-12**, No.4(1966), 425–430.

[11] Bezrukov S.L. *An Isoperimetric Problem* (in Russian), in Diskretny Analiz 43, Novosibirsk 1984, 1–16.

[12] Bezrukov S.L. *On an Isoperimetric Problem* (in Russian), in Proc. 6th All-Union Conf. Teor. Cybern. part 1, Saratov 1986, 25–28.

[13] Bezrukov S.L. *Specification of All Solutions of the Discrete Isoperimetric Problem which Have a Critical Cardinality* (in Russian), Doklady AN SSSR, **289**, No.3(1984), 520–524.

[14] Bezrukov S.L. *On the Minimization of the Boundary of Subsets in Hamming Space* (in Russian), in Algebraicheskie i Kombinatornye Metody v Prikladnoi Matematike, Gorky 1985, 45–48.

[15] Bezrukov S.L. *Specification the Maximal Sized Subsets of the Unit Cube with Respect to Given Diameter* (in Russian), Problemy Peredachi Informacii, **XXIII**, No.1(1987), 106–109.

[16] Bezrukov S.L. *On Constructing of Solutions of the Isoperimetric Problem in Hamming Space*, Math. USSR Sbornik, **63**, No.1(1989), 81–96.

[17] Bezrukov S.L. *Encoding of Analog Signals for Discrete Binary Channel*, in Proc. Int. Conf. ACCT-1, Varna 1988, 12–16.

[18] Bezrukov S.L. *The Kruskal-Katona Type Theorems for Posets*, to appear.

[19] Bezrukov S.L. *An Isoperimetric Problem for Manhattan Lattices*, in Proc. Int. Conf. Sets, Graphs and Numbers, Budapest 1991, 2–3.

[20] Bezrukov S.L., Grünwald N., Weber K. *On Optimal Edge Numberings of the n-Cube Graph*, Discr. Appl. Math. (to appear).

[21] Bezrukov S.L., Voronin V.P. *Extremal Ideals of the Lattice of Multisets with Respect to Symmetric Functionals* (in Russian), Diskretnaya Matematika, **2**, No.1(1990), 50–58.

[22] Bollobás B. *Martingals, Isoperimetric Inequalities and Random Graphs*, in Combinatorics: 7th Hungarian Colloq. Comb. Eger, July 5–10, 1987, Amsterdam etc. 1988, 113–139.

[23] Bollobás B., Radcliffe A.J. *Isoperimetric Inequalities for Faces of the Cube and the Grid*, Europ. J. Combinatorics, **11**(1990), 323–333.

[24] Bollobás B., Leader I. *Exact Edge-isoperimetric Inequalities*, Europ. J. Combinatorics, **11**(1990), 325–340.

[25] Bollobás B., Leader I. *Compressions and Isoperimetric Inequalities*, J. Comb. Th., **A 56**(1991), 47–62.

[26] Bollobás B., Leader I. *Isoperimetric Inequalities and Fractional Set Systems*, J. Comb. Th., **A 56**(1991), 63–74

[27] Bollobás B., Leader I. *Edge-Isoperimetric Inequalities in the Grid* (to appear).

[28] Bollobás B., Leader I. *An Isoperimetric Inequality for Independent Subsets of the Cube* (to appear).

[29] Bollobás B., Leader I. *An Isoperimetric Inequality on the Discrete Torus*, SIAM J. Discr. Math., **3**, No.1(1990), 32–37.

[30] Chukhrov I.P. *On Maximal Size of the Boundary of an Antichain* (in Russian), Diskretnaya Matematika, **1**, No.4(1989), 78–85.

[31] Clements G.F. *Sets of Lattice Points which Contain a Maximal Number of Edges*, Proc. Amer. Math. Soc., **27**, No.1(1971), 13–15.

[32] Clements G.F., Lindström B. *A Sequence of* ± 1*-Determinants With Large Values*, Proc. Amer. Math. Soc., **16**, No.3(1965), 548–550.

[33] Frankl P. *A New Short Proof for the Kruskal-Katona Theorem*, Discr. Math., **48**(1984), 327–329.

[34] Frankl P. *A lower Bound on the Size of a Complex Generated by an Antichain*, Discr. Math., **76**, No.1(1989), 51–56.

[35] Frankl P., Füredi Z. *A Short Proof for a Theorem of Harper about Hamming-Spheres*, Discr. Math., **34**(1981), 311–313.

[36] Füredi Z., Griggs J.R. *Families of Finite Sets with Minimum Shadows*, Combinatorica, **6**, No.4(1986), 355–363.

[37] Harper L.H. *Optimal Assignment of Numbers to Vertices*, J. Sos. Ind. Appl. Math, **12**, No.1(1964), 131–135.

[38] Harper L.H. *Optimal Numberings and Isoperimetric Problems on Graphs*, J. Comb. Theory, **1**, No.3(1966), 385–393.

[39] Hart S. *A Note on the Edges of the n-Cube*, Discr. Math., **14**(1976), 157–163.

[40] Karachanjan V.M. *A Discrete Isoperimetric Problem on Multidimensional Torus* (in Russian), Doklady AN Arm. SSR, **LXXIV**, No.2(1982), 61–65.

[41] Katona G.O.H. *The Hamming-Sphere Has Minimum Boundary*, Studia Scient. Math. Hungarica, **10**(1975), 131–140.

[42] Kleitman D.J. *On a problem of Yuzvinsky on Separating the n-Cube*, Discr. Math., **60**(1986), 207–213.

[43] Kleitman D.J. *Extremal Hypergraph Problems*, in London Math. Soc. Lect. Note Ser., Cambridge Univ. Press Cambridge, No.38(1979), 44–65.

[44] Kleitman D.J., Krieger M.M., Rothschild B.L. *Configurations Maximizing the Number of Pairs of Hamming-Adjacent Lattice Points*, Studies in Appl. Math., **L**, No.2(1971), 115–119.

[45] Korshunov A.D., Sapozhenko A.A. *On the Number of Binary Codes with Distance 2* (in Russian), in Problemy Kibernetiki 40, Nauka, Moscow 1983, 111–130.

[46] Kostochka A.V. *On Maximal Cardinality of the Boundary of a Filter in the Unit Cube* (in Russian), in Diskretny Analiz 40, Novosibirsk 1984, 49–61.

[47] Kostochka A.V. *An Upper Bound for the Cardinality of an Antichain in the Unit Cube* (in Russian), Diskretnaya Matematika, **1**, No.3(1989), 53–61.

[48] Körner J., Wei V.K. *Odd and Even Hamming Spheres Also Have Minimum Boundary*, Discr. Math., **51**(1984), 147–165.

[49] Körner J., Wei V.K. *Addendum to "Odd and Even Hamming Spheres Also Have Minimum Boundary"*, Discr. Math., **62**(1986), 105–106.

[50] Kruskal J.B. *The Number of Simplices in a Complex*, in Mathematical Optimization Techniques, Univ. of California Press, Berkeley, Calif. 1963, 251–278.

[51] Lindsey II J.H. *Assignment of Numbers to Vertices*, Amer. Math. Monthly, **7**(1964), 508–516.

[52] Lindström B. *The Optimal Number of Faces in Cubical Complexes*, Arkiv für Math., **8**, No.24(1971), 245–257.

[53] Lindström B., Zetterström *A Combinatorial Problem in the k-Adic Number System*, Proc. Amer. Math. Soc., **18**(1967), 166–170.

[54] Nigmatullin R.G. *Some Metric Relations in the Unit Cube* (in Russian), in Diskretny Analiz 9, Novosibirsk 1967, 47–58.

[55] Nigmatullin R.G. *Variational Principle in Boolean Algebra* (in Russian), in Diskretny Analiz 10, Novosibirsk 1967, 69–89.

[56] Sapozhenko A.A. *On the Number of Connected Subsets in Bipartite Graphs with Respect to Given Size of the Boundary* (in Russian), in Diskretny Analiz 45, Novosibirsk 1987, 42–70.

[57] Tiersma H.J. *A Note on Hamming Spheres*, Discr. Math., **54**(1985), 225–228.

[58] Torosjan B.E. *Multicomponemtal Solutions of the Discrete Isoperimetric Problem* (in Russian), Diskretnaya Matematika, (to appear).

[59] Wang D-.L., Wang P. *Discrete Isoperimetric Problems*, SIAM J. Appl. Math., **32**, No.4(1977), 860–870.

[60] Wang D-.L., Wang P. *Extremal Configurations on a Discrete Torus and a Generalization of the Generalized Macaulay Theorem*, SIAM J. Appl. Math., **33**, No.1(1977), 55–59.