On Edge Numberings of the $n-$cube Graph

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Abstract

Given two disjoint groups U and V each containing 2^{n-1} people. Suppose an experiment (or a game) requires pairwise comparisons between $u \in U$ and $v \in V$ in such a way that $u \in U$ $(v \in V)$ is compared exactly with n people $v \in V$ $(u \in U)$. Only one comparison can be done at a time, and the comparisons are made along the $n2^{n-1}$ edges of the $n-$cube graph Q_n with vertex classes U and V. The paper deals with sequential orderings of these $n2^{n-1}$ pairs $(u, v) \in U \times V$ for which the maximum time duration a person has to stay in the sequence for comparison purposes or the total time duration of the 2^n people in the sequence, respectively, is small.

Analogous problem of finding optimal sequences of all $k-$subsets of an $n-$set or of all pairs $(x, y) \in X \times Y$, $|X| = |Y| = n$, were considered in [7] and [3] respectively.

1 Introduction

Given two disjoint groups U and V each containing 2^{n-1} people. Suppose an experiment (or a game) requires pairwise comparisons between $u \in U$ and $v \in V$ in such a way that $u \in U (v \in V)$ is compared exactly with n people $v \in V$ $(u \in U)$. Suppose further only one comparison can be done at a time. For a sequential ordering f of the $n2^{n-1}$ comparison pairs $e = (u, v) \in U \times V$ denote by $f(e)$ the position of e in this sequence. If $E(x)$ means the set of all pairs e containing the person x, then

$$D_f(x) = \max_{e,e' \in E(x)} |f(e) - f(e')| + 1$$

represents the time duration the person x has to stay in the sequence for comparison purposes. The problems considered are to find sequences f that minimize the maximum stay duration

$$W_f = \max_{x \in U \cup V} D_f(x),$$

or the total stay duration

$$S_f = \sum_{x \in U \cup V} D_f(x)$$

respectively.

Suppose we take the $n-$cube graph Q_n in order to model the prescribed situation. Q_n consists of the 2^n vertices $x = (x_1, ..., x_n)$, $x_i \in \{0, 1\}$, and the $n2^{n-1}$ edges between vertices differing in exactly one coordinate. A vertex $x \in Q_n$ is called even (odd) if it contains an even (resp., odd) number of ones. Then we assume that the groups U

and V are represented by the vertex classes of even (resp. odd) vertices of Q_n and the comparisons are made along the edges of Q_n. I.e. we are concerned with edge numberings (one-to-one correspondences $f : E(Q_n) \mapsto \{1, 2, ..., n2^{n-1}\}$).

We may prove the optimality of certain numberings f in the sense that they minimize W_f or S_f respectively only for small values of n ($n \leq 4$; cf. Section 5). On the general case we have only bounds for the parameters

$$S(n) = \min_f S_f$$

(Section 3) and

$$W(n) = \min_f W_f$$

(Section 4), where both minima are taken over all $(n2^{n-1})!$ numberings of the edges of Q_n.

Finally, in Section 6 we show that a random numbering yields asymptotically the worst case (i.e. $2^n \cdot n2^{n-1}$ or $n2^{n-1}$ respectively) with high probability.

2 Further notations and basic facts

G : also the vertex set of the graph G,
$|G|$: the order of G = number of vertices,
$E(G)$: the edge set of G.

For vertices $x = (x_1, ..., x_n)$ of Q_n

$$\begin{aligned} |x| &= \sum_{i=1}^{n} x_i, \\ \nu(x) &= \sum_{i=1}^{n} x_i \cdot 2^{n-1}, \end{aligned}$$

L_v : lexicographical ordering of the vertices of Q_n; $x <_{L_v} y$ iff $\nu(x) < \nu(y)$,
H : Harper ordering of the vertices of Q_n; $x <_H y$ iff $|x| < |y|$ or $|x| = |y|$ and $\nu(x) > \nu(y)$,
L_e : lexicographical ordering of the edges if Q_n; for $e = (x, y)$, $|x| < |y|$ and $e' = (x', y')$, $|x'| < |y'|$ we write $e <_L e'$ iff $\nu(y) < \nu(y')$ or $y = y'$ and $\nu(x) > \nu(x')$.

For an edge set $A \subseteq E(Q_n)$:
$V(A)$: set of all vertices incident with edges from A,

$$\begin{aligned} m(t) &= \min_{|A|=t} |V(A)|, \\ M(t) &= \max_{|A|=t} |V(A)|. \end{aligned}$$

Fact 1 (Lindström [8]). *The minimum $m(t)$ is achieved for the first t edges of L_e.*

$G_1(A)$: set of edges $e \in E(Q_n) \setminus A$ adjacent to edges of A,
$R_1(A)$: subset of vertices of $V(A)$ incident with edges of $G_1(A)$,

$$r_1(t) = \min_{|A|=t} |R_1(A)|.$$

For a vertex set $X \subseteq Q_n$:
$E(X)$: set of all edges e induced by X (both ends of e are in X),
$\Gamma(X)$: set of all edges $e = (x, y)$, $x \in X$, $y \in Q_n \setminus X$ (edge boundary),
$R(X)$: set of all vertices of X incident with edges of $\Gamma(X)$ (vertex boundary),
$I(X) = X \setminus R(X)$: set of all interior vertices,

$$\begin{aligned} e(m) &= \max |E(X)|, \\ \gamma(m) &= \min |\Gamma(X)|, \\ r(m) &= \min |R(X)|, \\ i(m) &= \max |I(X)|, \end{aligned}$$

where maxima and minima are taken over all $X \subseteq Q_n$ with $|X| = m$.

Fact 2 (Harper, Bernstein, Hart; cf.[4,6]). *The maximum $e(m)$ and simultaneously the minimum $\gamma(m)$ is achieved for the first m vertices of L_v. Moreover,*

$$e(m) \leq m \cdot \lceil \log_2 m \rceil / 2. \tag{1}$$

Fact 3 (Harper [5]). *The minimum $r(m)$ is achieved for the first m vertices of H.*

Fact 4 *The minimum of $|G_1(A)|$, taken over all edge sets $A \subseteq E(Q_n)$ of cardinality t, is achieved for the first t edges of L_e.*

Proof.
Let $|A| = t$ be given and suppose $|V(A)| = m$. Then $|G_1(A)| = |E(V(A))| - t + |\Gamma(V(A))|$ and $2|E(V(A))| + |\Gamma(V(A))| = nm$. Hence,

$$|G_1(A)| = nm - t - |E(V(A))| \geq nm - t - e(m).$$

Since $e(m+1) - e(m) \leq n$ (cf. Fact 2), Fact 1 implies that

$$\min_{|A|=t} |G_1(A)| = nm(t) - t - e(m(t)), \tag{2}$$

and the proof follows. ∎

For an edge numbering f :
$G_f(t)$: graph consisting of the edges with numbers $1, 2, ..., t$ and the vertices incident with these edges,
$\Theta_f(t)$: number of those vertices of $G_f(t)$ which are incident with an edge having a number $s > t$,
$\Theta(t) = \min_f \Theta_f(t)$.
(Clearly, $\Theta(t) = r_1(t)$, $t = 1, 2, ..., n2^{n-1}$).

Fact 5

$$W(n) \geq \max_{1 \leq t \leq n2^{n-1}} \min_{\substack{A \subseteq E(Q_n) \\ |A|=t}} |G_1(A)| + 1.$$

Proof.
Let f be an arbitrary edge numbering. Put $A = E(G_f(t))$ and define $l_1 = \max_{e \in G_1(A)} f(e)$. Then $l_1 \geq t + |G_1(A)|$ and therefore $W_f \geq l_1 - t + 1 \geq |G_1(A)| + 1$. ∎

For a numbering f and a vertex $x \in Q_n$:

$$\begin{aligned} \mu_f(x) &= \min_{e \in E(x)} f(e), \\ M_f(x) &= \max_{e \in E(x)} f(e). \end{aligned}$$

(Clearly, $D_f(x) = M_f(x) - \mu_f(x) + 1$).

Fact 6 *For any numbering f*

$$S_f = \sum_{t=1}^{n2^{n-1}} \Theta_f(t) + 2^n. \tag{3}$$

Proof.
A fixed vertex $x \in Q_n$ is counted by $\Theta_f(t)$ for $t = \mu_f(x)$, $\mu_f(x) + 1, ..., M_f(x) - 1$, i.e. exactly $M_f(x) - \mu_f(x)$ times. ∎

3 Bounds for $S(n)$

Theorem 1 *There are positive constants c_1, c_2 such that*

$$c_1 4^n \leq S(n) \leq c_2 \sqrt{n} \cdot 4^n.$$

Proof.
Lower bound: We are going to show that $S_f \geq c_1 4^n$ for an arbitrary edge numbering f of Q_n. Since for an arbitrary edge set set $A \subseteq E(Q_n)$ the equality $R_1(A) = R_1(E(Q_n) \setminus A)$ is evident from (4) we derive

$$S_f \geq \sum_{t=1}^{n2^{n-1}} \Theta_f(t) = \sum_{t=1}^{n2^{n-1}} r_1(t) \geq 2 \cdot \sum_{i=1}^{(n-1)2^{n-2}} r_1(t).$$

Now let $|A| = t$, $|V(A)| = m$ and $p = m - |R_1(A)|$ denotes the number of interior vertices (vertices of degree n in the graph $(V(A), A)$). Then $pn + |R_1(A)| \leq 2t$. Thus

$$|R_1(A)| \geq \frac{mn - 2t}{n-1}$$

and consequently

$$r_1(t) \geq \frac{m(t)n - 2t}{n-1}. \tag{4}$$

By (1) we know that $l\lceil \log_2 l \rceil/2 < t \leq (l+1)\lceil \log_2 (l+1) \rceil/2$ implies $m(t) \geq l+1$. Thus by (5)

$$S(n) \geq 2 \sum_{t=1}^{(n-1)2^{n-2}} r_1(t) \geq 2 \sum_{t=1}^{(n-1)2^{n-2}} \frac{m(t)n - 2t}{n-1} \geq 2 \left[\sum_{t=1}^{(n-1)2^{n-2}} m(t) - \frac{2}{n-1} \sum_{t=1}^{(n-1)2^(n-2)} t \right].$$

Easy computations show that

$$\frac{2}{n-1}\sum_{t=1}^{(n-1)2^{n-2}} t = n4^{n-2} - 4^{n-2} + O(2^n) \quad \text{and}$$

$$\sum_{t=1}^{(n-1)2^{n-2}} m(t) \geq \sum_{l=1}^{2^{n-1}-1} \left(\frac{l+1}{2}\lceil \log_2(l+1)\rceil - \frac{l}{2}\lceil \log_2 l\rceil\right)(l+1) =$$

$$= n4^{n-2} - \frac{2}{3}4^{n-2} + O(n2^n).$$

Hence

$$S(n) \geq \frac{2}{3}4^{n-2} + O(n2^n).$$

Upper bound: The following numbering f_1 yields $S_{f_1} \leq c_2\sqrt{n}\cdot 4^n$: Label the vertices of Q_n by $1, 2, ..., 2^n$ corresponding to the Harper ordering H, i.e. the $m-$th vertex of this ordering is labeled by number m. Put $f_1((1,2)) = 1$. Let all edges induced by vertices $1, 2, ..., m$ be numbered by $1, 2, ..., t$ and let the vertex $m+1$ in Q_n be joined to vertices $i_1 < i_2 < ... < i_j < m+1$. Then set $f_1((i_1, m+1)) = t+1$, $f_1((i_2, m+1)) = t+2, ..., f_1((i_j, m+1)) = t+j$. (The numbering f_1 is depicted in Fig.1 for Q_5.)

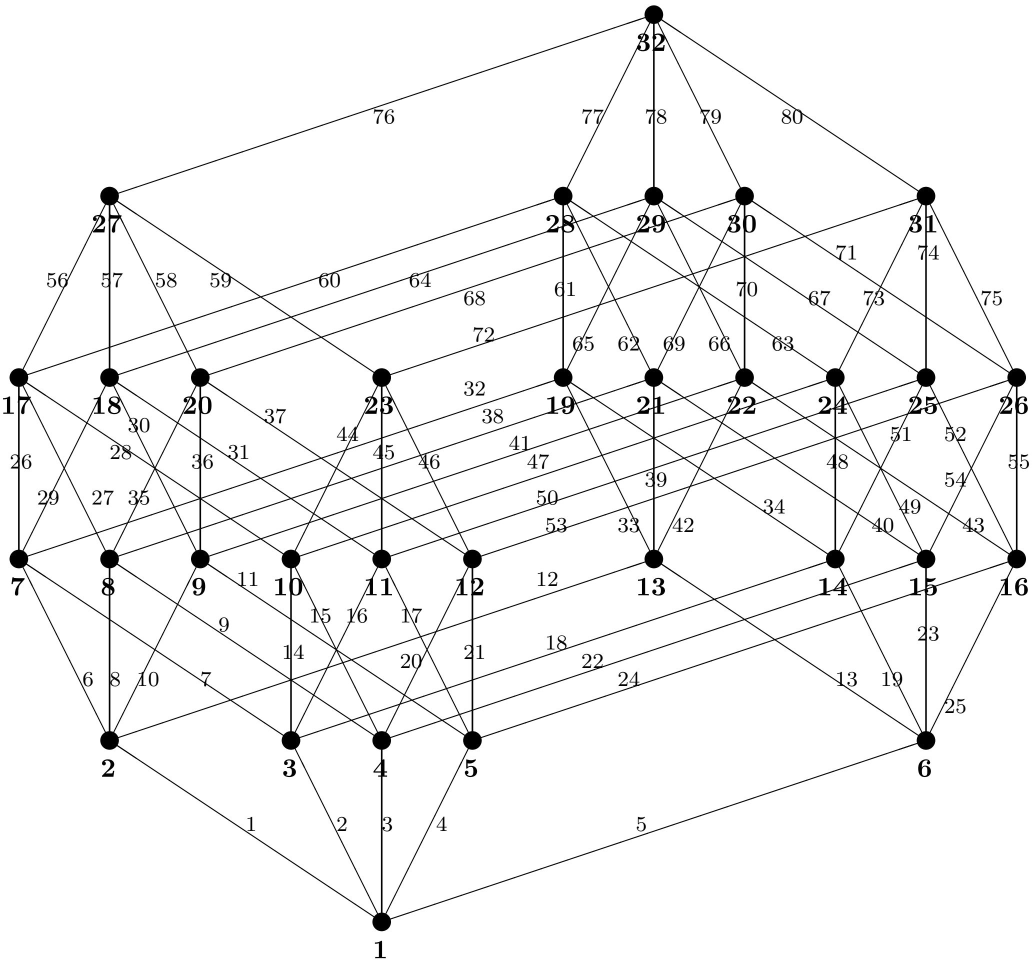

Fig.1

Now, by definition of f_1, $\sum_{i=0}^{l} i\binom{n}{i} < t \leq \sum_{i=0}^{l+1} i\binom{n}{i}$ implies

$$\sum_{i=0}^{l}\binom{n}{i} < |G_{f_1}(t)| \leq \sum_{i=0}^{l+1}\binom{n}{i}$$

and $\Theta_{f_1}(t) \leq \binom{n}{l} + \binom{n}{l+1}$. Hence

$$\sum_{t=1}^{n2^{n-1}} \Theta_f(t) \leq \sum_{l=0}^{n-1} \sum_{\sum_{i=0}^{l} i\binom{n}{i} < t \leq \sum_{i=0}^{l+1} i\binom{n}{i}} \left(\binom{n}{l} + \binom{n}{l+1}\right) =$$

$$= \sum_{l=0}^{n-1}(l+1)\binom{n}{l+1}\left(\binom{n}{l} + \binom{n}{l+1}\right) \sim 2 \cdot \sum_{l=1}^{n} l\binom{n}{l}^2 = n\binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} \cdot \sqrt{n} \cdot 4^n.$$

The estimation shows that indeed every "level by level" numbering f of the edges of Q_n yields the upper bound $n\binom{2n}{n}$. ("Level by level" means a numbering of the edges (x, y), $|x| = l$, $|y| = l+1$, by numbers $\sum_{i=0}^{l} i\binom{n}{i} + 1, ..., \sum_{i=0}^{l+1} i\binom{n}{i}$, $l = 0, 1, ..., n-1$). ∎

It is easy to verify that S_{f_1} has the order of magnitude $\sqrt{n}\cdot 4^n$: If $\sum_{i=0}^{l} i\binom{n}{i} < t \leq \sum_{i=0}^{l+1} i\binom{n}{i}$ then $\sum_{i=0}^{l}\binom{n}{i} < |G_{f_1}(t)| \leq \sum_{i=0}^{l+1}\binom{n}{i}$, and from [9] we know that $\Theta_{f_1}(t) \geq \binom{n}{i}$. Therefore,

$$S_{f_1} \geq \sum_{t=1}^{n2^{n-1}} \Theta_f(t) \geq \sum_{l=0}^{n-1} \sum_{\ldots<t\leq\ldots} \binom{n}{l} = \sum_{l=0}^{n-1}(l+1)\binom{n}{l+1}\binom{n}{l} \sim \frac{1}{2\sqrt{\pi}}\cdot\sqrt{n}\cdot 4^n.$$

Thus, also a more careful calculation of the upper bound of S_{f_1} does not help to improve the order of the upper bound in the Theorem.

4 Bounds for $W(n)$

Theorem 2 *There is a positive constant c_3 such that for $n \geq 3$*

$$2^{n-1} + 2^{n-3} + 1 \leq W(n) \leq c_3\sqrt{n}\cdot 2^n.$$

Proof.
Lower bound: Put $\varphi(n,t) = nm(t) - t - e(m(t))$ and $\varphi(n,t) = \max_{1\leq t\leq n2^{n-1}} \varphi(n,t)$ and $\varphi'(n) = \max\varphi(n,t)$ over all t such that $t = e(m(t))$. Then by (3) and Fact 4, $W(n) \geq \varphi(n) + 1 \geq \varphi'(n) + 1$. But $\varphi'(m) \geq 2^{n-1} + 2^{n-3}$ is easily shown by induction. Clearly, by Fact 1, $\varphi(3,2) = 5$, $e(2) = e(3) = 2$ and so $\varphi(3) \geq 2^2 + 2^0$. Now let $A \subseteq E(Q_n)$ consists of the first t edges in the lexicographical ordering L_e, where the t is chosen such that $t = e(m(t))$ and $|G_1(A)| = \varphi(n,t) = \varphi'(n)$. Then $V(A)$ consists of the first $m(t)$ vertices of L_v. Adding an $(n+1)-$th coordinate 0 (1) to all vertices $x \in V(A)$ we get the vertex sets X_0 (resp. X_1) of Q_{n+1}. Clearly $X_0 \cup X_1$ contains the first $2m(t)$ vertices of L_v in Q_{n+1}, and the edge set $B = E(X_0 \cup X_1)$ contains the first $2e(m(t)) + m(t)$ edges of L_e in Q_{n+1}. Thus, we have $|B| = e(m(|B|))$ and $|G_1(B)| = \varphi(n+1, |B|) = 2|G_1(A)| = 2\varphi'(n)$, and consequently $\varphi'(n+1) \geq 2\varphi'(n)$.
Upper bound: Let $x \in Q_n$, $|x| = k$. For the numbering f_1 (cf. the proof of Theorem 1) we have

$$M_{f_1}(x) - \mu_{f_1}(x) + 1 \leq$$

$$\sum_{i=1}^{k+1} i\binom{n}{i} - \left(\sum_{i=1}^{k-1} i\binom{n}{i} + 1\right) + 1 = k\binom{n}{k} + (k+1)\binom{n}{k+1} \leq 2\left\lceil\frac{n}{2}\right\rceil\binom{n}{\lfloor n/2\rfloor}.$$

Note again that every "level by level" numbering f of the edges of Q_n yields the same upper bound. Moreover, it is easy to see that W_{f_1} has the order of magnitude $\sqrt{n}\cdot 2^n$ such that a more careful calculation of W_{f_1} does not help to improve the order of the upper bound in the Theorem. ∎

Remark 1 *From (2) and Fact 1 we obtain $\varphi(5,17) = 21 = 2^4 + 2^2 + 2^0$, where $17 = e(m(17)) = e(11)$. Thus, the same reasoning as in the proof of the lower bound in Theorem 2 yields $W(n) \geq 2^{n-1} + 2^{n-3} + 2^{n-5} + 1$ for $n \geq 5$. Moreover, $\varphi(4,t) = 11$ with $t = e(m(t))$ is not possible.*

Remark 2 *Instead of the trivial lower bound (3) it holds the following generalization:*

$$W(n) \geq \max_{1 \leq t \leq 2^{n-1}} \min_{\substack{A \subseteq E(Q_n) \\ |A|=t}} \max_{1 \leq k \leq n-1} \left(\frac{1}{k} \sum_{i=1}^{k} |G_i(A)| \right) + 1, \tag{5}$$

where $G_0(A) = A$, $G_1(A)$ *is defined in Section 2 and* $G_i(A) = G_1(G_{i-1}(A)) \setminus G_{i-2}(A)$, $i = 2, 3, ..., n-1$.

Proof.
Let be given a numbering f and denote $A = E(G_f(t))$ and $t_i = \max_{e \in G_t(A)} f(e)$, $i = 1, ..., k$. Then

$$W_n(f) \geq \max\{|t_1 - t|, |t_2 - t_1|, ..., |t_k - t_{k-1}|\} + 1,$$

and at least one of these differences is greater or equal to $\sum_{i=1}^{k} |G_i(A)|/k$, since $\max_{1 \leq i \leq k} t_i$ is at least $t + \sum_{i=1}^{k} |G_i(A)|$. For $k = 1$ we get (3). ∎

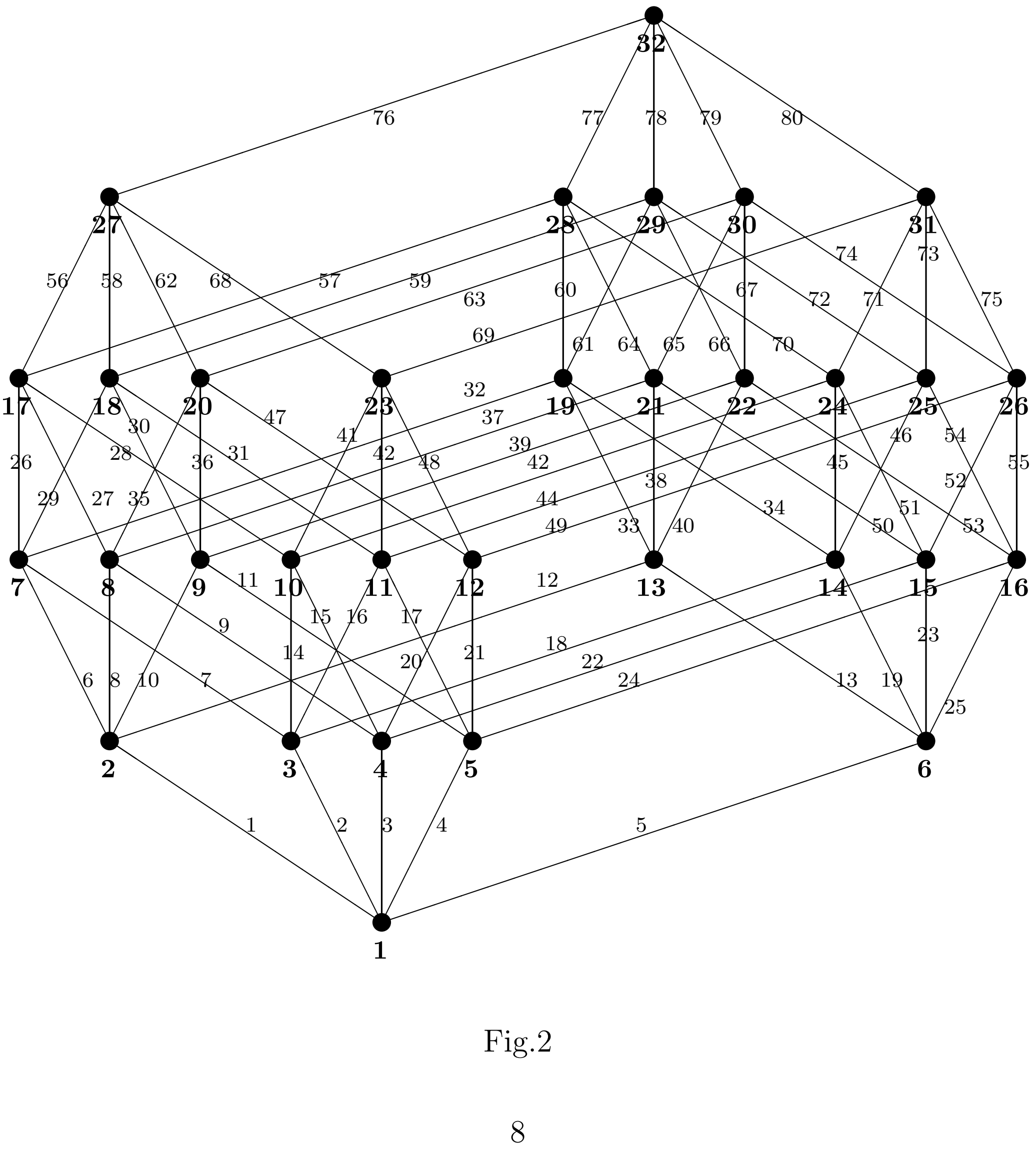

Fig.2

Remark 3 *For $n \geq 4$ the following numbering f_2 yields a slight improvement of f_1: As for f_1 the vertices of Q_n are labeled by $1, 2, ..., 2^n$ corresponding to the Harper numbering H. But now we number the edges "from below and from above": For $k = 2, 3, .., 2^n$ do*
a) Let A_0 consists of all unnumbered edges (i, k), $i < k$, and set $u_{k-1} = |A_0|$. Let A consists of all unnumbered edges $(2^n - k + 1, j)$, $j > 2^n - k + 1$. Let $A_1 = A \setminus A_0$, and set $v_{k-1} = |A_1|$.
b) Number the edges of A_0 in natural order (i.e. $f_2((i_1, k)) < f_2((i_2, k))$ if $i_1 < i_2$) beginning with the lowest possible integer, say $t_{k-1} + 1$ until $t_{k-1} + u_{k-1}$.
c) Number the edges of A_1 in natural order (i.e. $f_2((2^n - k + 1, j_1)) > f_2((2^n - k + 1, j_2))$ if $j_1 > j_2$) beginning with the highest possible integer, say $n2^{n-1} - s_{k-1}$ until $n2^{n-1} - s_{k-1} + v_{k-1} + 1$.
(See Fig.2 for the case $n = 5$). But obviously also W_{f_2} has the order of magnitude $\sqrt{n} \cdot 2^n$.

5 Small values

In this Section we show the correctness of the following Table 1.

n	2	3	4
$S(n)$	10	44	186
$W(n)$	3	7	14

Table 1.

Values of $S(n)$ **:**
for every edge numbering f we have

$$S_{f_1} = \sum_{t=1}^{n2^{n-1}} \Theta_f(t) + 2^n = \sum_{t=1}^{n2^{n-2}-1} \Theta_f(t) + \Theta(n2^{n-2}) + 2^n =$$

$$= \sum_{t=1}^{n2^{n-2}} r_1(t) + r_1(n2^{n-2}) + 2^n.$$

We want to find for every t, $1 \leq t \leq n2^{n-2}$ a set A of t edges of Q_n which minimizes $|R_1(A)| = r_1(t) = \Theta(t)$.
$n = 2$: It is easy to verify that the following Table 2.1 is correct

$\vert A\vert = t$	1	2
$\Theta(t)$	2	2

Table 2.1

Therefore we have $S_f(2) \geq 10$ for every edge numbering f. On the other hand for the numbering f_1 of Q_2 we have $S_{f_1}(2) = 10$. ∎

For an edge set $A \subseteq Q_n$ with $|A| = t$ and $|V(A)| = m$ we have the following inequality:

$$2t \geq n \cdot i(t) + r(t) \quad \text{and} \quad i(t) + r(t) = m. \tag{6}$$

$n = 3$: For an edge set A and the vertex set $V(A)$ of Q_3 we get with Fact 1 and Fact 3 the second and the third rows of Table 3.1 and 3.2 ($m(t)$, $M(t)$, $i(m)$, $r(m)$). Now we are going to prove the correctness of the $\Theta(t)$ in Table 3.1.

$\|A\| = t$ 1	2	3	4	5	6
$m(t)$ 2	3	4	4	5	6
$M(t)$ 2	4	6	8	8	8
$\Theta(t)$ 2	3	3	4	4	4

Table 3.1

$\|V(A) = m$	1	2	3	4	5	6	7	8
$i(m)$	0	0	0	1	1	2	4	8
$r(m)$	1	2	3	3	4	4	3	0

Table 3.2

For $1 \leq t \leq 3$ it is easy to check that $\Theta(t)$ is correct.
$t = 4$. From Table 3.1 we get $4 \leq |V(A)| \leq 8$. If $|V(A)| = 4$ and $r(|V(A)|) = 3$ then one of the four vertices must be an interior one. It follows that the four vertices induce only 3 edges in contradiction with $t = 4$. If $|V(A)| \geq 7$ and $|A| = 4$, then we can't have an interior vertex (see (7)) and therefore $|R_1(A)| \geq 7$. If $|V(A)| \in \{5, 6\}$ then we get from Table 3.2 $r(|V(A)|) = 4$ and it follows $\Theta(4) \geq 4$.
$t = 5$. From Table 3.1 we get $5 \leq |V(A)| \leq 8$. If $|V(A)| \geq 7$ and $|A| = 5$ then we have at most one interior vertex (see (7)) and therefore $|R_1(A)| \geq 6$. If $|V(A)| \in \{5, 6\}$ then we get from Table 3.2 $r(|V(A)|) = 4$ and it follows $\Theta(5) \geq 4$.
$t = 6$. From Table 3.1 we get $6 \leq |V(A)| \leq 8$. If $|V(A)| \geq 7$ and $|A| = 6$ then we have at most two interior vertices (see (7)) and therefore $|R_1(A)| \geq 5$. If $|V(A)| = 6$ then we get from Table 3.2 $r(|V(A)|) = 4$ and it follows $\Theta(6) \geq 4$.

Therefore $S(3) \geq 2(2 + 3 + 3 + 4 + 4) + 4 + 2^3 = 44$. On the other hand for the edge numbering f_1 of Q_3 we get $S_{f_1}(3) = 44$ and so $S(3) = 44$ (see Fig.3). ∎

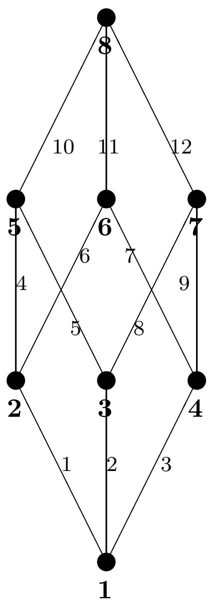

Fig.3

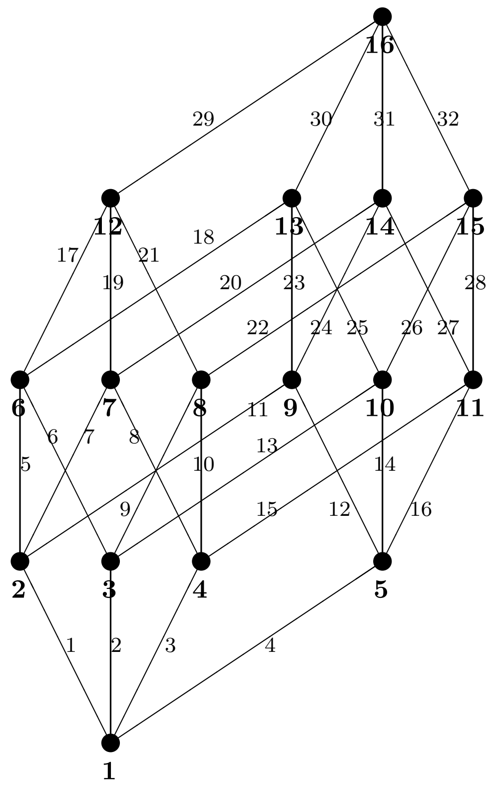

Fig.4

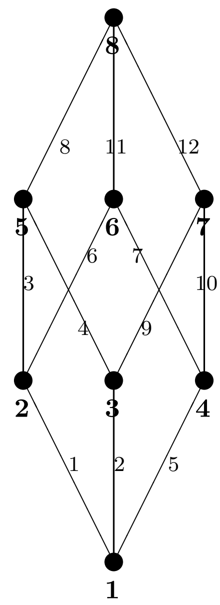

Fig.5

$n = 4$: For an edge set A and the vertex set $V(A)$ of Q_4 we get with Fact 1 and Fact 3 the second and the third rows of Table 4.1 and 4.2 ($m(t)$, $M(t)$, $i(m)$, $r(m)$). Now we are going to prove the correctness of the $\Theta(t)$ in Table 4.1.

$\|A\| = t$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$m(t)$	2	3	4	4	5	6	6	7	7	8	8	8	9	10	10	11
$M(t)$	2	4	6	8	10	12	14	16	16	16	16	16	16	16	16	16
$\Theta(t)$	2	3	4	4	5	5	6	6	6	6	7	7	7	7	7	6

Table 4.1

$\|V(A) = m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$l(m)$	0	0	0	0	1	1	1	2	2	3	5	5	6	8	11	16
$r(m)$	1	2	3	4	5	5	6	6	7	7	6	7	7	6	4	0

Table 4.2

For $1 \leq t \leq 4$ it is easy to check that $\Theta(t)$ is correct.
$t = 5$. From Table 4.1 we get $5 \leq |V(A)| \leq 10$. If $|V(A)| = 5$ and $r(|V(A)|) = 4$ then one of the five vertices must be an interior one. It follows that the five vertices induce only 4 edges in contradiction with $t = 5$. For $5 \leq |V(A)| \leq 10$ we get from Table 4.2 $r(|V(A)|) \geq 5$ and therefore $\Theta(5) \geq 5$.

$t=6$. From Table 4.1 we get $6 \le |V(A)| \le 12$ and then from Table 4.2 $r(|V(A)|) \ge 5$. therefore $\Theta(6) \ge 5$.
$t=7$. With the same arguments as for $t=5$ we get $\Theta(7) \ge 6$.
$t=8$. From Table 4.1 we get $7 \le |V(A)| \le 16$. For $|A|=8$ and $|V(A)| \ge 15$ we have no interior vertex (see (7)) and therefore $|R_1(A)| \ge 15$. If $7 \le |V(A)| \le 14$ we get from Table 4.2 $r(|V(A)|) \ge 6$ and it follows $\Theta(8) \ge 6$.
$t=9,10$. With the same arguments as for $t=8$ we get $\Theta(9) \ge 6$ and $\Theta(10) \ge 6$.
$t=11$. In [1] it is shown that $\Theta(11) \ge 7$.
$t=12$. From Table 4.1 we get $8 \le |V(A)| \le 16$. If $|V(A)|=8$ or 11, we get with the same argument as for $t=11$ $|R_1(A)| \ge 7$. If $|V(A)| \ge 14$ we have for $|A|=12$ at most three interior vertices (see (7)) and therefore $|R_1(A)| \ge 11$. If $|V(A)| \in \{9,10,12,13\}$ we get from Table 4.2 $r(|V(A)|)=7$ and it implies $\Theta(12) \ge 7$.
$t=13,14,15$. With the same arguments as for $t=12$ we get $\Theta(13) \ge 7$, $\Theta(14) \ge 7$ and $\Theta(15) \ge 7$.
$t=16$. From Table 4.1 we get $11 \le |V(A)| \le 16$. If $|V(A)| \ge 14$ we have for $|A|=16$ at most 6 interior vertices (see (7)) and therefore $|R_1(A)| \ge 8$. If $|V(A)| \in \{11,12,13\}$ we get from Table 4.2 $r(|V(A)|) \ge 6$ and it implies $\Theta(16) \ge 6$.

Therefore $S(4) \ge 2(2+3+4+4+5+5+6+6+6+6+7+7+7+7+7)+6+2^4 = 186$. On the other hand for the edge numbering f_2 of Q_4 we get $S_{f_2}(4)=186$, therefore $S(4)=186$ (see Fig.4). ∎

Remark 4 *Notice without proof that* $S(5)=782$ *and neither* f_1 *nor* f_2 *is optimal numbering. Figure 6 shows an optimal numbering for this case.*

Values of $W(n)$.
$n=2$. It is simple to verify that $W(2)=2$.
$n=3$. We will apply Theorem 2 and obtain $W(3) \ge 6$. Figure 5 shows an edge numbering f of Q_3 with $W_f(3)=6$ and therefore $W(3)=6$.
$n=4$. Let f be an edge numbering with $W_f(n)=W(n)$. We have the lower bound

$$W_f(n) \ge \max_{k,t} \left\lceil \sum_{i=1}^{k} |G_i(A)|/k \right\rceil + 1, \tag{7}$$

where A is the set of edges with $f(A)=\{1,2,...,t\}$.

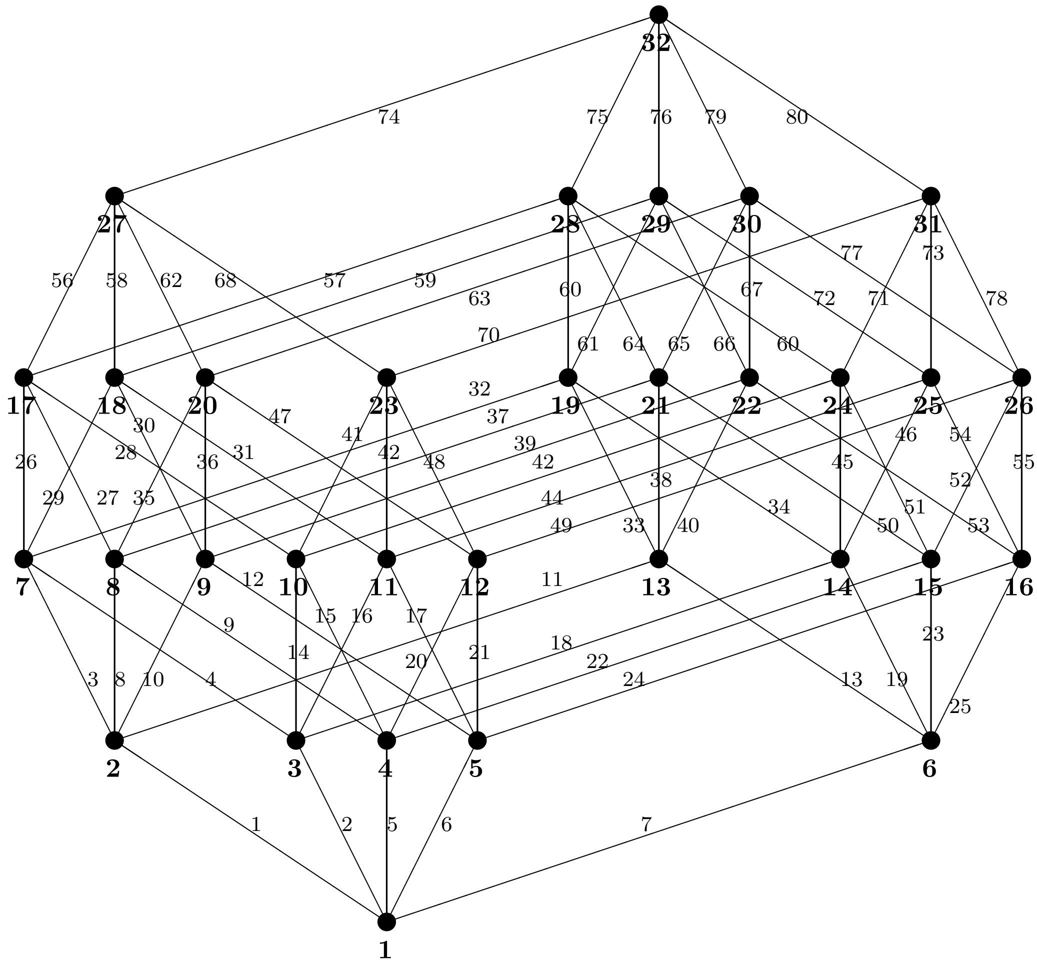

Fig.6

Let us consider the case $k = 2$ and $t = 3$. So $A = \{e_1, e_2, e_3\}$ with $f(A) = \{1, 2, 3\}$. Assume that there are two edges e_1 and e_2 without a common vertex. Let us divide Q_4 into two Q_3 with e_1 in one of the halves and e_2 in the other. For e_i, $i = 1, 2$ and the corresponding Q_3 we get $|G_1(e_i)| + |G_2(e_i)| = 10$. Since $G_1(e_1)$ covers 6 vertices in it's subcube we get

$$|G_1(A)| + |G_2(A)| \geq |G_1(e_1, e_2)| + |G_2(e_1, e_2)| - 1 \geq$$
$$2(|G_1(e_1)| + |G_2(e_1)| - 1 + 6 = 25.$$

Therefore $W(4) \geq 14$.

Assume now that all three edges e_1, e_2 and e_3 have a common vertex. In this case $|G_1(A)| + |G_2(A)| = 25$ and $W(4) \geq 14$ too. Figure 4 shows the edge numbering f_2 of Q_4 with $W_{f_2}(4) = 14$, therefore $W(4) = 14$. ∎

Remark 5 *Another proof of $W(4) \geq 14$ is based on the following idea. We use (8) with $k = t = 2$ and the fact that the reverse numbering of an optimal edge numbering of Q_n is also optimal.*

6 Random numberings

In this Section we consider the mean value of W_f and S_f respectively over all $(n2^{n-1})!$ possible edge numberings of Q_n. Taking the uniform probability distribution, $\text{Prob}(f) = 1/(n2^{n-1})!$ for all f, the parameters $\mu(x)$, $M(x)$, $D(x)$ (each for fixed vertex x), W and S are random variables. Then the expectation

$$\mathbf{E}S = \frac{1}{(n2^{n-1})!} \cdot \sum_f S_f$$

may be computed as follows

$$\mathbf{E}S = \mathbf{E}\left(\sum_x D(x)\right) = \sum_x \mathbf{E}D(x) = 2^n \mathbf{E}D = 2^n(\mathbf{E}M - \mathbf{E}\mu + 1)$$

since the random variables $D(x)$, $M(x)$ and $\mu(x)$ do not depend on x. Since for an arbitrarily fixed vertex $x \in Q_n$ with $E(x) = \{e_1, ..., e_n\}$ the lengths of the intervals $I_1, I_2, ..., I_{n+1}$ (cf. Fig.7) are identically distributed random variables these lengths have the same expectation $E = \frac{n2^{n-1}-n}{n+1}$. But $\mathbf{E}\mu = 1 + E$ and $\mathbf{E}M = n2^{n-1} - E$. Consequently

$$\mathbf{E}S = 2^n \cdot (n2^{n-1} - 2E) \sim n2^{2n-1}.$$

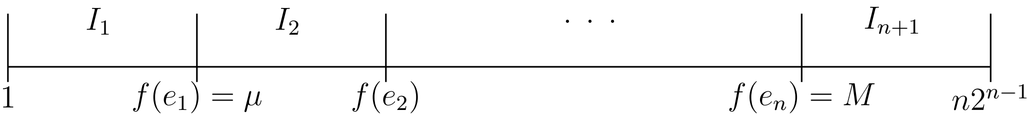

Fig. 7

Furthermore, for the expectation

$$\mathbf{E}W = \frac{1}{(n2^{n-1})!} \cdot \sum_f W_f$$

we have

$$\mathbf{E}W \geq \mathbf{E}D = \mathbf{E}M - \mathbf{E}\mu + 1 = n2^{n-1} - 2\mathbf{E} \sim n2^{n-1}.$$

Hence $\mathbf{E}W \sim n2^{n-1}$ since $W_f \leq n2^{n-1}$ is a trivial upper bound for all numberings f. We derived the following

Theorem 3 $\mathbf{E}S \sim n2^{2n-1}$, $\mathbf{E}W \sim n2^{n-1}$ *as* $n \mapsto \infty$. ∎

Since moreover $S_f \leq n2^{2n-1}$ and $W_f \leq n2^{n-1}$ for all numberings f, we have the following

Remark 6 *There are* $\epsilon = \epsilon(n)$ *and* $\delta = \delta(n)$ *both tending to* 0 *as* $n \mapsto \infty$, *such that for more than* $(1-\epsilon)(n2^{n-1})!$ *numberings* f *the inequalities* $W_f \geq (1-\delta)n2^{n-1}$ *and* $S_f \geq (1-\delta)n2^{2n-1}$ *are fulfilled.* ∎

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