# Encoding of Analog Signals for Binary Digital Channel

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**Abstract**

We present here an encoding procedure for ordered numbers in order to minimize the mean magnitude error of a signal, caused by transmission through a binary channel, where only $t \leq n$ fixed positions of $n$-words may be disturbed. It is shown that our code is optimal for the case when the probability of error is small enough.

## 1 Introduction

Suppose we have to send send each of $2^n$ numbers $k_1, ..., k_{2^n}$ through a binary channel. For example, we may assume that these numbers were taken from the output of an analog to code digital converter, and so we have to assign numbers $k_i$ to each vector of the $n$-cube. It is assumed that only single errors are likely in a transmitted word, and that $n-t$ fixed positions of a word ($0 \leq t \leq n$) are error-free and the other $t$ positions may be disturbed with probability $p$. If the vector assigned to $k_i$ was transmitted and the vector assigned to $k_j$ was received, then let $\Delta_{ij}^d = |k_i - k_j|$ denotes the absolute value of the error. Our main goal is to find the assignment so, that the average absolute error in transmission is minimized under the condition that the choice of the $2^n$ numbers $k_i$ is equally probable.

If $t = n$ and $\{k_1, ..., k_2 n\} = \{1, ..., 2^n\}$, then such a problem was solved in [3]. In [4] one can find a solution with $t = n$ and arbitrary $k_i$. It was shown that if $k_1 \leq \cdots \leq k_{2^n}$ then it the both mentioned cases the number $k_i$ should be assigned with the binary expansion of $i-1$. It was proved in [1] that if $p$ is small enough, then this encoding procedure is optimal. In our paper we found the optimal procedure for $t < n$.

## 2 Computation of the mean magnitude error

Assume that $k_1 \leq \cdots \leq k_{2^n}$. Call the vector assigned to $k_i$ *the i-th vector* and the $t$ error possible positions *the admissible positions*. Analogously to [3,4] let $r_i$ be the number of vectors, assigned to numbers $k_i$, which are neighbors in admissible positions to the $i$-th vector, when only the first $i$ numbers have been assigned. It follows that in the computation of $\sum_{i,j} \Delta_{ij}$, $k_i$ will have a coefficient $r_i - (t - r_i) = 2r_i - t$. Hence, the average value of a single error equals

$$(t \cdot 2^{n-1})^{-1} \cdot \sum_{i=1}^{2^n} (2r_i - t) \cdot k_i.$$

Consequently, the mean magnitude error $E$ equals

$$E = t \cdot p \cdot (1-p)^{t-1} \cdot (t \cdot 2^{n-1})^{-1} \cdot \sum_{i=1}^{2^n} (2r_i - t) \cdot k_i.$$

Using in this sum only the first degree on $p$ terms, we conclude that if $p$ is sufficiently small then

$$E = p \cdot 2^{-(n-1)} \cdot \sum_{i=1}^{2^n} (2r_i - t) \cdot k_i.$$

Therefore, the minimizing of $E$ is equivalent to the minimizing of $S = \sum_{i=1}^{2^n} r_i \cdot k_i$. Since the numbers $k_i$ form a nondecreasing sequence, it is easy to see that $S$ may be written in the form

$$S = \sum_{i=1}^{2^n} r_i \cdot \sum_{i=1}^{2^n} k_i - \sum_{m=1}^{2^n} a_m \cdot \sum_{i=1}^{m} r_i,$$

where all $a_m$, $1 \leq m \leq 2^n$, are some nonnegative numbers. Since $\sum_{i=1}^{2^n} r_i = t \cdot 2^{n-1}$, then for minimizing $S$ it is sufficient to maximize $\sum_{i=1}^{m} r_i$ for every $m = 1, ..., 2^n$, But $\sum_{i=1}^{m} r_i$ equals the number of admissible connections between numbered vectors when the first $m$ vectors have been numbered.

Consequently, for minimizing $E$ it is sufficient to numerate the vertices of $n$-cube (1 to $2^n$) so, that for every $m$, $1 \leq m \leq 2^n$, the subset consisting of the first $m$ numbered vectors should have a maximal number of admissible connections. Then in order to get the optimal code we should assign $k_i$ to the $n$-tuple, numbered by $i$.

# 3 Maximizing the number of admissible connections

It was proved in [3] that if $t = n$ then we may use the natural numbering of $n$-tuples with respect to the lexicographic order. Moreover, such a numbering is unique up to isometric transformations of the cube. Let us denote by $F_m^n$ (by $L_m^n$) the collection of the first (resp., the last) $m$ vertices of $B^n$ in the lexicographic order. Let $A \subseteq B^n$ and $R(A)$ denotes the number of of *"interior"* connections in $A$ and let $G(A)$ be the number of connections between vertices of $A$ and $B^n \setminus A$. Then

$$R(A) = n + |A| - G(A) \quad \text{and} \quad G(A) = G(B^n \setminus A). \tag{1}$$

Let us represent the integer $m$ in the form $m = p \cdot 2^t + q$, $0 \leq q < 2^t$, and split $B^n$ into $t$-subcubes by the nonadmissible positions. Call these subcubes *admissible subcubes.* Denote by $\tilde{A}$ the subset, consisting of $p$ arbitrary admissible subcubes united with the set $F_q^t$ in the $(p+1)$-th admissible subcube. Similar construction was used in [2] for an isoperimetric type problem.

**Theorem 1** *$\tilde{A}$ has the maximal possible number of admissible connections among all $m$-element subsets of $B^n$.*

*Proof.*

Let $A \subseteq B^n$ and let $B_i$ be an admissible subcube. Denote $A_i = A \cap B_i$. Then the number of admissible connections in $A$ equals $\sum_{i=1}^{2^{n-t}} R(A_i)$. Let us replace $A_i$ by $F^t_{|A_i|}$ in each admissible subcube. For the obtained set $C$ we get

$$\sum_{i=1}^{2^{n-t}} R(C_i) \geq \sum_{i=1}^{2^{n-t}} R(A_i),$$

where $C_i = C \cap B_i$. Let $B_i$ and $B_j$ be a pair of admissible subcubes.

If $|A_i| + |A_j| \leq 2^t$, then replace $A_j$ to $A_j^1 = L^t_{|A_j|}$. Then $R(A_j) = R(A_j^1)$. Project now $A_j^1$ into $B_i$. We obtain the subset $A_j^2$ in such a way and $A_i \cap A_j^2 = \emptyset$, $R(A_j^1) = R(A_j^2)$. Replacing $A_i \cup A_j^2$ to $F^t_{|A_i|+|A_j|}$, we obtain the subset $A_i^1$ in the subcube $B_i$ and by [3] $R(A_i^1) > R(A_i) + R(A_j)$.

If $|A_i| + |A_j| > 2^t$, then consider the complements of $A_i$ and $A_j$ in the corresponding $t$-subcubes. Then $|\overline{A_i}| + |\overline{A_j}| < 2^t$ and after a projection of $\overline{A_j}$ into $B_i$ we derive similarly to above that

$$G(\overline{A_i}) + G(\overline{A_j}) < G(F^t_{|\overline{A_i}|+|\overline{A_j}|-2^t}).$$

Considering again the complements of the obtained sets in $B_i$, $B_j$ and taking into account the equality $R(P^t_m) = R(L^t_m)$ and (1), we get

$$R(A_i) + R(A_j) < R(B_j) + R(F^t_{|A_i|+|A_j|-2^t}).$$

After a finite number of such operations we construct the desired set $\tilde{A}$ from $A$. ∎

Now in order to specify our numeration let us order arbitrarily the admissible subcubes $B_i$. Let $\tilde{\alpha}, \tilde{\beta} \in B^n$ and $\tilde{\alpha} \in B_i$, $\tilde{\beta} \in B_j$. We say that $\tilde{\alpha}$ *precedes* $\tilde{\beta}$ if the subcube $B_i$ precedes $B_j$ or $B_i = B_j$ and $\tilde{\alpha}$ precedes $\tilde{\beta}$ in order, which is isomorphic to the lexicographic order (in $t$ dimensions). Let us number the $n$-tuples in this order.

It is easy to show that all numberings maximizing the number of admissible connections may be formed in such a way.

# References


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