Some New Results on Macaulay Posets

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Dedicated to Rudolf Ahlswede on his 60th birthday

Abstract

Macaulay posets are posets for which there is an analogue of the classical Kruskal-Katona theorem for finite sets. These posets are of great importance in many branches of combinatorics and have numerous applications. We survey mostly new and also some old results on Macaulay posets. Emphasis is also put on construction of extremal ideals in Macaulay posets.

1 Introduction

Macaulay posets are, informally speaking, posets for which an analogue of the classical Kruskal-Katona theorem for finite sets holds. They are related to many other combinatorial problems like isoperimetric problems on graphs [9] (see also section 3) and problems arising in polyhedral combinatorics. Several optimization problems can be solved within the class of Macaulay posets, or at least for Macaulay posets with additional properties (cf. section 5). Therefore, Macaulay posets are very useful and interesting objects.

A few years ago, the classical Macaulay posets listed in section 2 were the only known essential examples, and, consequently, the theory of Macaulay posets was more or less the theory of these examples. In his book [30, chapter 8], Engel made a first attempt for unification the theory of Macaulay posets. Although the book appeared quite recently, a number of new examples, relations and applications have been found meantime. In this paper, our objective is to give a survey on Macaulay posets that includes these new results and updates [30].

We start with some basic facts and definitions in section 1 and the classical examples in section 2. For all definitions not included here we refer to Engel's book [30]. In section 3 we proceed with constructions for Macaulay posets and relations to isoperimetric problems. New examples of Macaulay posets are presented in section 4. Section 5 is devoted to optimization problems on Macaulay posets.

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1.1 Some basic definitions

Let P be a partially ordered set (briefly, poset) with the associated partial order $\leq$. For $x, y \in P$, we say that y *covers* x, denoted by $x \lessdot y$, if $x \leq y$ and there is no $z \in P$ such that $z \neq x, y$ and $x \leq z \leq y$. An *antichain* is defined as a subset $X \subseteq P$ such that the conditions $x, y \in X$ and $x \leq y$ imply $x = y$.

A subset $X \subseteq P$ is an *ideal* (or *downset*) if the conditions $x \in X$ and $y \leq x$ imply $y \in X$. If X is an antichain, then the set $I(X) := \{y \in P \mid y \leq x \text{ for some } x \in X\}$ is an ideal, which is called *ideal generated by* X. Conversely, if I is an ideal, then the set $\max(I) := \{x \in I \mid x \not\leq y \text{ for any } y \in I,\ y \neq x\}$ is an antichain, which is called the *set of maximal elements* of I.

A *rank function* on P is a function $r : P \mapsto I\!N$ such that $r(x) = 0$ for some minimal element x of P and $r(y) = r(z) - 1$ whenever $y \lessdot z$. The poset P is called *ranked*, if a rank function on P exists. The *rank* of P is defined by $r(P) := \max\{r(x) \mid x \in P\}$, where $r(P) = \infty$ is allowed. A ranked poset P is called *graded* if all minimal elements have rank 0, and all maximal elements have rank $r(P)$.

The *dual* P^* of P is the poset on the same set of elements with the partial order defined by: $x \leq^* y$ iff $y \leq x$. If P is ranked with $r(P) < \infty$, then P^* is ranked. If P is ranked with $r(P) = \infty$, then P^* is not ranked in the usual sense. In this case $r^*(x) := -r(x)$ will considered to be the rank function for P^*.

If P is ranked, then the set $\{x \in P \mid r(x) = i\}$ is called the i-th *level* of P and is denoted by $N_i(P)$ or P_i. The *(lower) shadow* of an element $x \in P_i$ is the set $\Delta(x) := \{y \in P \mid y \lessdot x\}$, and its *upper shadow* is $\nabla(x) := \{y \in P \mid x \lessdot y\}$. The lower shadow $\Delta(X)$ (resp. upper shadow $\nabla(X)$) of a subset $X \subseteq P_i$ is defined as the union of the lower (resp. upper) shadows of its elements. For given integers i and m with $1 \leq i \leq r(P)$ and $1 \leq m \leq |P_i|$, the *shadow minimization problem* (SMP) consists in finding an m-element subset $X \subseteq P_i$ such that $|\Delta(X)| \leq |\Delta(Y)|$ for all $Y \subseteq P_i$ with $|Y| = m$. We say that a subset $X \subseteq P_i$ is *optimal* if it has minimum shadow among all subsets of P_i of the same size. Obviously, the SMP is at least NP-hard, since it implies a solution to the Minimum Cover Problem.

The *(cartesian) product* $P \times Q$ of two posets P and Q is the set of all pairs (x, y) with $x \in P$, $y \in Q$, where the partial order is given by: $(x, y) \leq_{P \times Q} (x', y')$ iff $x \leq_P x'$, $y \leq_Q y'$. If P and Q are ranked, then the poset $P \times Q$ is ranked too, and the rank function for $P \times Q$ is given by: $r(x, y) := r_P(x) + r_Q(y)$. The n-th *(cartesian) power* of a poset P is the poset $P^n := P \times P \times \cdots \times P$ (n times).

1.2 Macaulay posets

Let P be a ranked poset and consider some total order $\preceq$ of its elements. Note that we do not claim the order $\preceq$ to be a *linear extension* of P. For a subset $X \subseteq P$ and a natural number $m \leq |X|$ we will use the notation $C(m, X)$ (resp. $L(m, X)$) for the set of the first (resp. last) m elements of X w.r.t. $\preceq$. In particular, for $X \subseteq P_i$ we abbreviate $C(|X|, P_i)$ and $L(|X|, P_i)$ by $C(X)$ and $L(X)$, respectively. The operation of replacing $X \subseteq P_i$ with $C(X)$ is called *compression*, and we say that X is *compressed* if $X = C(X)$. Compressed subsets will also be called *initial segments* (IS), whereas a *final segment* of P_i is a subset $X \subseteq P_i$ with $X = L(X)$. A *segment* of P_i simply is a set of elements of P_i which are consecutive w.r.t. $\preceq$ (restricted to P_i). For an element $x \in P_i$, the initial segment of P_i whose last element w.r.t. $\preceq$ is x is denoted by $\mathcal{F}_i(x)$.

The poset P is said to be a *Macaulay poset* if there exists a total order $\preceq$ of its elements (called *Macaulay order*) such that

$$\Delta(C(X)) \subseteq C(\Delta(X)) \text{ for all } X \subseteq P_i \text{ and for all } i = 1, \ldots, r(P). \tag{1}$$

If (1) is satisfied for a ranked poset P with a partial order $\leq$ and for a total order $\preceq$ of the elements of P, then the triple $(P, \leq, \preceq)$ is called *Macaulay structure*.

It is easy to verify (cf, [30] for details) that (1) holds iff the conditions $\mathbf{N}_1$ and $\mathbf{N}_2$ given below are satisfied for all $X \subseteq P_i$ and for all $i = 1, \ldots, r(P)$:

$\mathbf{N}_1$: $|\Delta(C(X))| \leq |\Delta(X)|$,

$\mathbf{N}_2$: $C(\Delta(C(X))) = \Delta(C(X))$.

According to $\mathbf{N}_1$, compressed subsets are optimal for the Macaulay poset P. Therefore, $\mathbf{N}_1$ is called the condition of *nestedness* (of the optimal subsets). By $\mathbf{N}_2$, the shadow of a compressed set is a compressed set again. That is why $\mathbf{N}_2$ is said to be the condition of *continuity*.

For a total order $\preceq$ of the elements of P denote by $\preceq^*$ its inverse.

Proposition 1 (Bezrukov [8]). *$(P, \leq, \preceq)$ is a Macaulay structure iff so is $(P^*, \leq^*, \preceq^*)$.*

For many applications it turns out to be natural and useful to choose a Macaulay order rank greedily. We say that a total order $\preceq$ is *rank greedy* (on P), if it is a linear extension of the partial order $\leq$ (i.e. if $x \leq y$ implies $x \preceq y$), and if, in addition, $r(x) = r(y) + 1$ implies $x \preceq y$ whenever the last element of $\Delta(x)$ w.r.t. $\preceq$ precedes y in the order $\preceq$. It can be easily shown (see e.g. [30]) that for every Macaulay poset there exists a rank greedy Macaulay order of its elements. The proof for this and the next assertion can be found in [30].

Proposition 2 *If a total order $\preceq$ is rank greedy for a Macaulay poset P, then $\preceq^*$ is rank greedy for P^*.*

If we associate a rank greedy total order with some Macaulay poset P, then we also say that P is rank greedy. Note that all Macaulay orders presented in sections 2 and 4 are rank greedy.

1.3 The shadow function

Let P be a Macaulay poset. The *shadow function* sf_i assigns with each subset $X \subseteq P_i$ the number $sf_i(X) = |\Delta(C(X))|$. We briefly discuss some properties of the shadow function.

The lower and upper *new shadows* of an element $x \in P$ are defined by:

$$\begin{aligned}\Delta_{new}(x) &:= \{y \in P \mid y \lessdot x \text{ and there is no } z \in P \text{ with } z \preceq x,\ z \neq x,\ y \lessdot z\},\\ \nabla_{new}(x) &:= \{y \in P \mid x \lessdot y \text{ and there is no } z \in P \text{ with } x \preceq z,\ z \neq x,\ z \lessdot y\},\end{aligned}$$

respectively. Note that the upper new shadow of x in P is exactly the lower new shadow of x in P^*. The lower new shadow $\Delta_{new}(X)$ (resp. upper new shadow $\nabla_{new}(X)$) of a subset $X \subseteq P$ is the union of the lower (resp. upper) new shadows of its elements. The shadow function sf_i is called *additive* if the inequality

$$|\Delta_{new}(X)| \geq |\Delta_{new}(Y)| \geq |\Delta_{new}(Z)|$$

is satisfied for all segments $X, Y, Z \subseteq P_i$ with X being initial, Z being final, and $|X| = |Y| = |Z|$. We say that P is additive if sf_i is additive for all $i = 0, \ldots, r(P)$.

Proposition 3 (Engel [30]). *Let P be a Macaulay poset. P is graded and additive iff its dual P^* is graded and additive.*

The Macaulay poset P is called *shadow increasing* if for all $i = 0, \ldots, r(P) - 1$ and for any initial segments $X \subseteq P_i$ and $Y \subseteq P_{i+1}$ with $|X| = |Y|$ the inequality $|\Delta(X)| \leq |\Delta(Y)|$ holds. We say that P is *final shadow increasing* if we have $|\Delta_{new}(X)| \leq |\Delta_{new}(Y)|$ for all $i = 0, \ldots, r(P)-1$ and for any final segments $X \subseteq P_i$ and $Y \subseteq P_{i+1}$ with $|X| = |Y|$. Finally, P is said to be *weakly shadow increasing* if $|\Delta_{new}(X)| \leq |\Delta_{new}(Y)|$ holds for any segments $X \subseteq P_i$ and initial segments $Y \subseteq P_j$ such that $i \leq j$, $|X| = |Y|$ and $X \cup Y$ is an antichain.

Proposition 4 (Engel, Leck [31]). *Let P be a Macaulay poset.*

a. If P is final shadow increasing, then P^* is shadow increasing.

b. Let P be graded, additive, and shadow increasing. If P^* is shadow increasing, then P is final shadow increasing.

c. If P is a graded, additive and shadow increasing, then P is weakly shadow increasing.

2 Some known Macaulay posets

2.1 Boolean lattices

Boolean lattices are certainly the most popular examples of Macaulay posets. For a natural number n the Boolean lattice B^n is defined as the collection of all subsets of $[n] := \{1, 2, \ldots, n\}$ partially ordered by inclusion, i.e. $X \leq Y$ for $X, Y \subseteq [n]$ iff $X \subseteq Y$. The unique rank-function on B^n maps a set $X \subseteq [n]$ to $|X|$. Representing the subsets of $[n]$ by their characteristic vectors, it is obvious that B^n is isomorphic to the n-th cartesian power of the chain $0 \lessdot 1$ of length one.

The *lexicographic order* of the elements of B^n is defined by $X \preceq_{lex} Y$ iff $\max(X \setminus Y) \leq \max(Y \setminus X)$, where $\max(\emptyset) := 0$. The following theorem, which meantime became a classical one, was proved by Kruskal [39] and Katona [37].

Theorem 1 (Kruskal-Katona theorem). $(B^n, \subseteq, \preceq_{lex})$ *is a Macaulay structure.*

The solution to the SMP provided by Kruskal-Katona theorem is not unique, in general. However, for at least 2^{n-1} cardinalities m the IS of the lexicographic order of size m is essentially a unique optimal subset, as it is shown in the next theorem. Denote $\Delta(m,k) = |\Delta(C(m, B^n_k))|$.

Theorem 2 (Füredi, Griggs [32]). *If $\Delta(m+1,k) > \Delta(m,k)$ for some $k \geq 1$, then the set $C(m, B^n_k)$ is a unique optimal subset of size m (up to isomorphism).*

This result, however, is a corollary of more general results [7, 8] which concern the VIP. Without going into details, for which readers are referred to a survey [8], we mention another corollary of results on VIP.

Theorem 3 (Bezrukov [7]). *If $A \subseteq B^n_k$ is optimal for some $k \geq 0$, then so is for $\Delta(A)$.*

Presently it is not known if this property is valid for other Macaulay posets.

2.2 Chain products

Cartesian product of chains, called also *lattice of multichains*, is a well-studied generalization of Boolean lattices. For positive integers n and $k_1 \leq k_2 \leq \cdots \leq k_n$ the chain product $S(k_1, k_2, \ldots, k_n)$ consists of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that $x_i \in \{0, 1, \ldots, k_i\}$ for $i = 1, 2, \ldots, n$. The partial order is a coordinatewise one: $\mathbf{x} \leq \mathbf{y}$ iff $x_i \leq y_i$ for $i = 1, 2, \ldots, n$. Again we have a uniquely determined rank-function, namely $r(\mathbf{x}) = \sum_{i=1}^n x_i$. Obviously, $S(k_1, k_2, \ldots, k_n)$ is the cartesian product of the chains $0 \lessdot 1 \lessdot \cdots \lessdot k_i$, $i = 1, 2, \ldots, n$.

A natural extension of the lexicographic order to chain products is established by: $\mathbf{x} \preceq_{lex} \mathbf{y}$ iff $\mathbf{x} = \mathbf{y}$ or $x_j < y_j$, where j is the smallest index with $x_j \neq y_j$.

Theorem 4 (Clements-Lindström theorem). $(S(k_1, \ldots, k_n), \leq, \preceq_{lex})$ *is a Macaulay structure.*

A short proof of this theorem is based on shifting technique and is published in [41]. A principally different approach used in [17] for the MWI problem (cf. Section 5.2) implies a short proof too. The properties of chain products given in the following theorem are important for many applications (see section 5.1 for instance).

Theorem 5 (Clements [18]). *Chain products are additive and shadow increasing.*

2.3 The star posets

Another natural way to generalize Boolean lattices is to consider the chain $0 \lessdot 1$ as a star with just two vertices. This leads to cartesian products of stars. For positive integers n and $k_1 \leq k_2 \leq \cdots \leq k_n$ the star poset $T(k_1, k_2, \ldots, k_n)$ consists of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that $x_i \in \{k_n - k_i, k_n - k_i + 1, \ldots, k_n\}$ for $i = 1, 2, \ldots, n$, where the partial order is given by: $\mathbf{x} \leq \mathbf{y}$ iff $x_i = y_i$ or $y_i = k_n$ for $i = 1, 2, \ldots, n$. The unique rank-function on $T(k_1, k_2, \ldots, k_n)$ is given by $r(\mathbf{x}) = |\{i \mid x_i = k_n\}|$.

To introduce a Macaulay order $\preceq$ on $T(k_1, k_2, \ldots, k_2)$, define $\mathbf{x}(j) := \{i \in [n] \mid x_i = j\}$ for $\mathbf{x} \in T(k_1, k_2, \ldots, k_n)$ and $j = 0, 1, \ldots, k_n$. Now $\preceq$ is defined as follows: $\mathbf{x} \preceq \mathbf{y}$ iff $\mathbf{x} = \mathbf{y}$ or $\mathbf{y}(h) \prec_{lex} \mathbf{x}(h)$, where h is the smallest number with $\mathbf{x}(h) \neq \mathbf{y}(h)$.

Theorem 6 $(T(k_1, k_2, \ldots, k_n), \leq, \preceq)$ *is a Macaulay structure.*

This theorem is found by Lindström [48] for the case $k_1 = \cdots = k_n = 2$ (his proof, however, contains a gap), and is proved by Leeb [47] and Bezrukov [6] in the case $k_1 = \cdots = k_n$. Actually, both mentioned proofs can be extended for the case $k_1 \neq k_n$. Explicit proofs for this general case are given in [30, 42].

Theorem 7 *Star products are additive and shadow increasing.*

The additivity part of this theorem is due to Clements [20] (see [30] for simplification), the shadow increase property was shown by Leck [43] by using an idea of Kleitman.

2.4 Colored complexes

Obviously, for $k_n \geq 2$ the star product $T(k_1, k_2, \ldots, k_n)$ is not isomorphic to its dual. Engel [30] observed that the duals of star products are isomorphic to colored complexes which were introduced by Frankl, Füredi and Kalai [34] in the case $k_n - k_1 \leq 1$.

To define colored complexes in general, for positive integers n and $k_1 \leq k_2 \leq \cdots k_n$, and for $i = 1, 2, \ldots, n$, let the i-th *color class* be the set

$$A_i := \{i, n+i, 2n+i, \ldots, (k_i - 1)n + i\}.$$

Now the colored complex $Col(k_1, k_2, \ldots, k_n)$ consists of all subsets $X \subseteq A := \bigcup_{i=1}^{n} A_i$ such that $|X \cap A_i| \leq 1$ for $i = 1, 2, \ldots, n$, i.e. of all subsets of A which meet every color class at most once. The corresponding partial order is the usual set inclusion.

Due to the isomorphism mentioned above, Proposition 1 and Theorem 6, and, respectively, Proposition 3, yield the following corollaries.

Corollary 1 (Colored Kruskal-Katona theorem [34]). $(Col(k_1, k_2, \ldots, k_n), \subseteq, \preceq_{lex})$ *is a Macaulay structure.*

Corollary 2 *The colored complexes are additive.*

The following theorem is the result of yet another application of the Kleitman's idea mentioned above.

Theorem 8 (Leck [43]). *Colored complexes are shadow increasing.*

3 Construction of Macaulay posets

3.1 Posets with a given shadow function

Here we show that for any shadow function sf_i there exists a Macaulay poset with this shadow function. Obviously, it suffices to construct Macaulay posets with two levels only. Let P be a ranked poset with $r(P) = 1$ and consider the SMP on its top level P_1. Denote by $\Delta(m)$ the minimal size of the shadow of a set consisting of m elements of P_1. Obviously, the sequence $\{\Delta(m)\}$ is nondecreasing.

Proposition 5 *For any nondecreasing sequence $\{\Delta(1), ..., \Delta(p)\}$ there exists corresponding Macaulay poset P with $r(P) = 1$.*

To construct such a poset, denote $P_1 = \{a_1, \ldots, a_p\}$ and $P_0 = \{b_1, \ldots, b_{\Delta(p)}\}$. We define a partial order $\leq$ on $P = P_0 \cup P_1$ as follows. For any $i = 1, \ldots, p$ set $a_i > b_j$ for $j = 1, \ldots, \Delta(i)$. Obviously, the constructed poset is Macaulay and the labelings of a_i's and b_i's provide Macaulay orders on P_1 and P_0 respectively.

Similarly Macaulay posets with more levels can be constructed. This construction is, in a sense, invertible. Given a Macaulay poset $(P, \leq, \preceq)$, construct another poset $Q = (P, \sqsubseteq)$ as follows. Take an element $a \in P_i$ for some $i > 1$ and consider $\mathcal{F}_i(a)$. Then $\Delta(\mathcal{F}_i(a)) = \mathcal{F}_{i-1}(b)$ for some $b \in P_{i-1}$. Let $c \in \mathcal{F}_{i-1}(b)$ and assume $c \not\leq a$. Now we extend the partial order $\leq$ by setting $c \leq a$.

Proposition 6 (Bezrukov, Portas, Serra [16]). *The poset Q is Macaulay.*

3.2 Posets related to isoperimetric problems on graphs

Let $G = (V_G, E_G)$ be a graph. For $A \subseteq V_G$ denote

$$\begin{aligned} E(A) &= \{(u, v) \in E_G \mid u \in A,\ v \notin A\}, \\ E(m) &= \max_{|A|=m} |E(A)|. \end{aligned}$$

Consider an edge-isoperimetric problem (EIP): for any $m \leq |V_G|$ find $A \subseteq V_G$ such that $|A| = m$ and $|E(A)| = E(m)$. We say that the edge-isoperimetric problem has *nested* *solutions* if there exists a numbering of V such that each IS is an optimal set. For more information on edge-isoperimetric problems on graphs readers are referred to the survey [12].

Assume that the EIP has nested solutions for the graph G. We construct a Macaulay poset $(P, \leq)$ with $|P| = |V_G|$ by induction on $|V_G|$ (cf. [11]). If $|V_G| = 1$, then the poset is trivial. For $|V_G| > 1$ let $V_G = \{1, \dots, |V_G|\}$ and assume that for each $m = 1, \dots, |V_G|$ the subset $\{v_1, \dots, v_m\} \subseteq V_G$ is optimal. Note that for $m < |V_G|$ this subset is also optimal for the subgraph G' which is induced by the vertex set $\{1, \dots, |V_G| - 1\}$. Construct the representing poset $(P', \leq')$ for G' by induction. Now extend P' by adding a new element v at level $i = E(|V_G|) - E(|V_G| - 1)$ and extend the partial order $\leq'$ by setting v to be greater than any element of P' at level $i - 1$. This procedure results in the poset $(P, \leq)$.

Proposition 7 (cf. [11]). *A poset obtained according to the EIP-construction is Macaulay.*

What is interesting that if a poset P represents a graph G, and if P^n is Macaulay, then the EIP on G^n has nested solutions [9, 10]. The inverse proposition is, however, not correct, in general. However, the posets P^n are good candidates for being Macaulay (cf. the discussion in section 5.3).

Now we turn to a *vertex-isoperimetric problem* on $G = (V_G, E_G)$. For $A \subseteq V_G$ denote

$$\begin{aligned}\Gamma(A) &= \{v \in V_G \setminus A \mid (v, u) \in E_G,\ u \in A\},\\ \Gamma(m) &= \min_{|A|=m} |\Gamma(A)|.\end{aligned}$$

The vertex-isoperimetric problem (VIP) consists in finding for a given $m \leq |V_G|$ a set $A \subseteq V_G$ such that $|A| = m$ and $|\Gamma(A)| = \Gamma(m)$. Such problems often arise in combinatorics. For a survey we refer to [8].

We additionally assume that for any IS $A \subseteq V_G$ the set $A \cup \Gamma(A)$ is an IS, too. This property corresponds to the continuity in the definition of Macaulay poset and holds for many graph families.

Let $V_G = \{1, \dots, |V_G|\}$, where any IS represents an optimal set. We construct a poset $(P, \leq)$ with $r(P) = 1$ and $|P| = 2|V_G|$ as follows. Let $P_0 = \{b_1, \dots, b_{|V_G|}\}$ and $P_1 = \{a_1, \dots, a_{|V_G|}\}$. We set $b_i < a_i$ for $i = 1, \dots, |V_G|$. Furthermore, if $(i, j) \in E_G$, then set $b_i < a_j$ and $b_j < a_i$.

Proposition 8 *The poset obtained according to the VIP-construction from a graph G is Macaulay iff G satisfies the nestedness and continuity properties with respect to the VIP.*

3.3 Product theorems

Counterexamples show that if P and Q are Macaulay posets, then $P \times Q$ is not necessarily Macaulay. For example, if P is a poset whose Hasse diagram is isomorphic to $K_{p,p}$ for $p \geq 2$ (i.e. we have a special case of a so-called *complete* poset [28]) then $P \times P$ is not Macaulay in contradistinction to a conjecture in [28]. Indeed, if $m \leq p$, then a set of m elements of P_1^2 has minimal shadow iff these elements agree in some entry whose rank in P is 0. However, the shadow of any element of P_2^2 consists of $2p$ elements of P_1^2, which do not contain p elements of the form above.

Thus, a condition on P and Q is needed for a product theorem. The situation is, however, simple if Q is a trivial poset with $r(Q) = 0$. In this case a necessary and sufficient condition for P is found by Clements:

Theorem 9 (Clements [21]). *If $r(Q) = 0$, then $P \times Q$ is additive and Macaulay iff so is P.*

Probably, the next case in this hierarchy are posets of the form $P \times C_q$ with C_q being a chain with q elements. Counterexamples show that a condition on P is required for $P \times C_q$ to be Macaulay. However, this is not the case for $r(P) = 1$, as our result shows.

Theorem 10 *Let P be a poset with $r(P) = 1$ and let $q \geq 1$. Then $P \times C_q$ is a Macaulay poset iff P is Macaulay.*

3.4 A local-global principle

Consider the SMP on a cartesian power P^n of a Macaulay poset P. There exists a powerful technique for establishing the Macaulayness of such posets, which, in particular, involves induction on the number n of posets in the product. However, the general arguments within this technique work for $n \geq 3$ only. The case $n = 2$ is a special one and must be considered separately.

A similar situation also occurs in the *edge isoperimetric* problem on graphs (see section 3.3). Ahlswede and Cai proved in [1] that if the *lexicographic order* (see section 2) provides nestedness in EIP, then it is so for any $n \geq 3$. It turns out that the last result, which is called the *local-global principle* in [1], is valid for the edge-isoperimetric problem also with respect to some other total orders [12].

In what concerns the SMP, the above approach can not be directly applied because of the necessity to maintain the level structure of a poset. It turns out, however, that for the validity of such a principle with respect to the lexicographic order it is important that the poset satisfies some additional conditions, which have no analogies for graphs yet.

We call a Macaulay poset P *strongly Macaulay* if it is additive, shadow increasing and final shadow increasing. Note that Theorems 9 and 10 are valid with respect to strongly Macaulay posets too. Denote by $\mathcal{M}$ the class of ranked posets having only one maximum and only one minimum element.

Proposition 9 *A poset $P \in \mathcal{M}$ is strongly Macaulay iff so is its dual P^*.*

Theorem 11 (Bezrukov, Portas, Serra [16]). *Let $(P, \leq, \preceq) \in \mathcal{M}$ be strongly Macaulay and rank-greedy. Let the lexicographic order $\preceq^2$ be Macaulay for P^2. Then for any $n \geq 2$ the lexicographic order $\preceq^n$ is a Macaulay order for P^n.*

The assumptions concerning the poset P in Theorem 11 are essential, as the following result shows.

Theorem 12 (Bezrukov, Portas, Serra [16]). *Let $(P, \leq, \preceq)$ be a Macaulay poset. Furthermore, let $r(P) \geq 3$ and assume the orders $\preceq^2$ and $\preceq^3$ are Macaulay for P^2 and P^3, respectively. Then for any $n \geq 1$ one has: $P^n \in \mathcal{M}$, P^n is rank greedy, and P^n is strongly Macaulay.*

As an application of the local-global principle consider the following poset $(T(k), \leq) \in \mathcal{M}$ of rank k. For $1 \leq i \leq k-1$ the i^{th} level of $T(k)$ consists of two elements a_i and b_i. Denote by b_0 and a_k the elements of T_0 and T_k, respectively. The partial order is defined as follows: $x < y$ iff $r(x) < r(y)$. We define the total order $\preceq$ on $T(k)$ by setting $b_{i-1} \prec a_i$ for $i = 1, \ldots, k$ and $a_i \prec b_i$ for $i = 1, \ldots, k-1$. Obviously, the order $\preceq$ is Macaulay on $(T(k), \leq)$.

Theorem 13 (Bezrukov, Portas, Serra [16]). *For any $k \geq 1$ and any $n \geq 1$ the poset $(T^n(k), \leq_\times, \preceq^n)$ is Macaulay.*

Further posets for which the local-global principle is applicable can be constructed using Proposition 6. Let P satisfy the assumptions of Theorem 11, and construct the poset $Q = (P, \sqsubseteq)$ as in section 3.3. Then Theorem 11 is applicable to Q. Indeed, the poset Q is Macaulay by Proposition 6. Now consider P^2. Since

$$\Delta_{P^2}(\mathcal{F}_i((x,y))) = \{(x,\xi) \mid \xi \in \Delta_P(\mathcal{F}_{i-r_P(x)}(y))\} \cup \{(\xi,y) \mid \xi \in \Delta_P(\mathcal{F}_{i-r_P(y)}(x))\},$$

then $\Delta_{P^2}(\mathcal{F}_i((x,y)) = \Delta_{Q^2}(\mathcal{F}_i((x,y)))$. Therefore, if P satisfies the assumptions of Theorem 11, then so does Q. On the other hand, since the lexicographic order is Macaulay for P^2, then so it is for P^4, for example. Extending P^2 as shown in section 3.1 results in a new poset, for which Theorem 11 is applicable.

4 New Macaulay posets

In this section we present some further new families of Macaulay posets. We start with posets which are factorable by using the cartesian product operation in subsections 1 - 3 and proceed with two posets which do not appear to be cartesian products.

4.1 The products of trees and spider poset

Evidently, the classical Macaulay posets mentioned in Section 2 (we mean the Boolean lattice, the chain products, and the star poset) have something in common. Namely, the Hasse diagrams of the underlying posets in the product are trees. These posets are also *upper-semilattices*. For $a, b \in P$ denote by $\sup_P(a,b)$ an element $c \in P$ (if it exists) such that $a \prec c$, $b \prec c$ and $c \prec d$ if $a \prec d$ and $b \prec d$. The poset P is an *upper-semilattice* if for any $a, b \in P$, $\sup_P(a,b)$ exists and is unique.

Denote by $\mathcal{P}$ the class of upper semilattices P whose Hasse diagrams are trees. For which posets $P \in \mathcal{P}$ any their cartesian posers P^n are Macaulay ? Denote by $Q(k,l) \in \mathcal{P}$ the poset with the element set $\{0, 1, \dots, (k+1)l\}$, and the partial order $\leq$ being defined as follows: $\alpha \leq \beta$ iff (i) $\alpha = \beta \pmod{k+1}$ and $\alpha \leq \beta$, or (ii) $\beta = (k+1)l$. The Hasse diagram of $Q(k,l)$ is a regular *spider* with k legs consisting of l vertices each.

Theorem 14 (Bezrukov [10]). *Suppose for some poset $P \in \mathcal{P}$ that P^n is Macaulay for some integer $n \geq r(P) + 3$. Then P is isomorphic to $Q(k,l)$ for some $k \geq 1$ and $l \geq 1$.*

It turns out that the inverse theorem is also valid.

Theorem 15 (Bezrukov, Elsässer [15]). *The poset $Q^n(k,l)$ is Macaulay for all integers n, k and l.*

The Macaulay order for $Q^n(k,l)$ is quite complicated and involves, in particular, the star poset order. We refer readers to [15] for exact definitions.

Looking back at Theorem 6 for star posets it is natural to ask if all cartesian products of the form $Q(k_1,l) \times Q(k_2,l) \times \cdots \times Q(k_n,l)$ are Macaulay. We conjecture an affirmative answer. On the other hand, it is easily seen that products of the form $Q(k,l_1) \times Q(k,l_2) \times \cdots \times Q(k,l_n)$ are not Macaulay in general.

4.2 Generalized submatrix orders

Let n and $k_1 \leq k_2 \leq \dots k_m$ be positive integers such that $k_0 := n - \sum_{i=1}^m k_i \geq 0$. Furthermore, let $A_0, A_1, \dots, A_m$ be the sets defined by

$$\begin{aligned} A_0 &:= \{1, 2, \dots, k_0\}, \\ A_i &:= \left\{ \sum_{j=0}^{i-1} k_j + 1, \sum_{j=0}^{i-1} k_j + 2, \dots, \sum_{j=0}^{i} k_j \right\} \text{ for } i = 1, 2, \dots, m. \end{aligned}$$

Clearly, the sets A_i $(i = 0, 1, \dots, m)$ form a partition of $[n] = \{1, 2, \dots, n\}$.

The *generalized submatrix order* $\mathcal{S} := SM(n; k_1, \dots, k_m)$ consists of all subsets X of $[n]$ such that $A_i \not\subseteq X$ for all $i = 1, 2, \dots, m$. The corresponding partial order is given by: $X \leq Y$ iff $X \subseteq Y$. According to this definition, $\mathcal{S}$ is isomorphic to the cartesian product $B^{k_0} \times \tilde{B}^{k_1} \times \cdots \times \tilde{B}^{k_m}$, where $\tilde{B}^s$ denotes the Boolean lattice B^s without its maximal element.

The name generalized submatrix order refers to the work of Sali [51, 53] who actually considered the dual of $\mathcal{S}$ in the case $m = 2$, $k_0 = 0$. Sali proved for this poset several analogies to classical theorems on finite sets (Sperner, Erdös-Ko-Rado). For this poset, he also solved the problem of minimizing the number of atoms which are covered by an m-element subset of the i-th level for given i, m and conjectured Theorem 16 below in an equivalent form.

Theorem 16 (Leck [45, 46]). *$(\mathcal{S}, \subseteq)$ is a Macaulay poset.*

Before the above theorem was established, the closely related problem of finding ideals of maximum rank (cf. section 5.3) was solved by Vasta [54] for $\mathcal{S}^*$ with $k_0 = 0$. Using Theorem 16, a more general statement is now implied by Theorem 27.

In the proof of Theorem 16, again the case $m = 2$ required some special treatment, a modification of the well-known shifting operator for finite sets was used to settle this case. The following theorem is commonly used in the proof for $m > 2$, which is done by induction.

Theorem 17 (Leck [46]). *Generalized submatrix orders are additive.*

Another interesting poset which is related to the generalized submatrix orders is the poset M^n of square submatrices of a square matrix of order n ordered by inclusion. This poset also was studied by Sali [50, 52] with respect to Sperner and intersecting properties. For $n \leq 3$ the poset M^n is Macaulay, but not for $n \geq 4$ in contradistinction to a conjecture in [28].

4.3 The torus poset

Denote by T_k the poset whose Hasse diagram can be obtained from two disjoint chains of length k each by identifying their top and bottom vertices. Obviously, the Hasse diagram of T_k is a cycle of length $2k$.

Let $T^n_{k_1,\ldots,k_n} = T_{k_1} \times \cdots \times T_{k_n}$. The solution to the SMP for this poset follows from a solution to a more general problem: the VIP (cf. Section 3.2). In order to show the relation, let us consider a bipartite graph G. Fix a vertex $v_0 \in V_G$ and denote by G_i the set of all vertices of G at distance i from v_0. This leads to a ranked poset P with $P_i = G_i$ whose Hasse diagram is isomorphic to G. Assume that a solution to VIP on G satisfies the nestedness and continuity properties. Moreover, we assume that the total order $\mathcal{O}$ which provides a solution to the VIP orders the vertices of G_i in sequence. In other words, if A is an IS of $\mathcal{O}$ and $\sum_{i=0}^{r} |G_i| \leq |A| \leq \sum_{i=0}^{r+1} |G_i|$, then A contains a ball of radius r centered in v_0 and is contained in the ball of radius $r+1$ with the same center.

Obviously, a solution to the SMP with respect to the minimization of $\nabla(\cdot)$ for the subsets of P_r follows. Moreover, each IS of the order $\mathcal{O}$ restricted to P_r provides an optimal set. This problem is equivalent to the SMP with respect to the minimization of $\Delta(\cdot)$ for the dual of P. Thus, both P^* and P are Macaulay.

The Macaulay order for $T^n_{k_1,\ldots,k_n}$, thus, can be obtained from the VIP-order $\mathcal{T}$ for the torus. This order is first established in [36], mentioned in the survey [8] and recently rediscovered in [49] and the readers are referred to these papers for exact definitions.

Theorem 18 (Karachanjan [36], Riordan [49]). *Any IS of the $\mathcal{T}$-oder provides a solution to the VIP. Moreover, the $\mathcal{T}$-oder satisfies the continuity property.*

4.4 Subword orders

Let us now turn to a first example of a Macaulay poset which is not representable as a cartesian product of nontrivial factors.

Let $n \geq 2$ be an integer, and let Ω denote the set $\{0, 1, \dots, n-1\}$. In the sequel, we call Ω the *alphabet*. The *subword order* $SO(n)$ consists of all strings (called *words*) that contain symbols (called *letters*) from Ω only. The partial order on $SO(n)$ is the subword relation, i.e. we have $x_1 x_2 \dots x_k \leq y_1 y_2 \dots y_l$ iff there is a set $\{i_1, i_2, \dots, i_k\} \subseteq \{1, 2, \dots, l\}$ of indices such that $i_1 < i_2 < \dots < i_k$ and $x_j = y_{i_j}$ for $j = 1, 2, \dots, k$. In other words, $\mathbf{x} \leq \mathbf{y}$ holds iff the word $\mathbf{x}$ can be obtained from the word $\mathbf{y}$ by successively deleting letters. By this definition, the rank of an element of $SO(n)$ equals its length, that means $r(x_1 x_2 \dots x_i) = i$. The only element of $N_0(SO(n))$ is the *empty word* ε.

Consider the case $n = 2$. Clearly, the level $N_i(SO(2))$ consists of all 0-1-words of length i and, therefore, in an obvious way its elements can be considered as the elements of the Boolean lattice B^i. It was shown by Harper [35] that, among all subsets $X \subseteq B^i$ of fixed cardinality, the IS in the *VIP-order* minimizes $|\Gamma_B(X)|$ (the size of the vertex-boundary of X in the Boolean lattice B^i). This order induces a total order of the elements for each level of $SO(2)$. For convenience, we define $w(x_1 x_2 \dots x_i) := |\{j \mid x_j = 1, 1 \leq j \leq i\}|$. Now the rank greedy extension of the VIP-order to the whole poset $SO(2)$ is given by the following conditions:

(1) $\mathbf{x} \preceq_{vip} \mathbf{y}$ if $w(\mathbf{x}) < w(\mathbf{y})$,

(2) $\mathbf{x} \preceq_{vip} \mathbf{y}$ if $w(\mathbf{x}) = w(\mathbf{y})$ and there is some $j \leq \min\{r(\mathbf{x}), r(\mathbf{y})\}$ such that $x_j > y_j$ and $x_h = y_h$ for $h = 1, 2, \dots, j-1$,

(3) $\mathbf{x} \preceq_{vip} \mathbf{y}$ if $w(\mathbf{x}) = w(\mathbf{y})$, $r(\mathbf{x}) \leq r(\mathbf{y})$ and $x_j = y_j$ for $j = 1, 2, \dots, r(\mathbf{x})$.

The next theorem reflects the importance of the VIP-order.

Theorem 19 (Ahlswede, Cai [2], Daykin, Danh [24, 25], Bezrukov [9]). *$(SO(2), \leq, \preceq_{vip})$ is a Macaulay structure.*

Let us remark that there are also several other Macaulay orders for $SO(2)$ which are specified by Daykin [29].

Based on the numerical approach of Ahlswede and Cai in [2], Engel and Leck [31] provided a relatively simple proof of Theorem 19. One of the main observations relates the SMP for $SO(2)$ to the VIP for Boolean lattices: If $X \subseteq N_i(SO(2))$ is a final segment, then $|\nabla(X)| = |\Gamma_B(X)| + 2|X|$ holds. Another interesting observation is that $C(X)$ and $L(X)$ are isomorphic for any $X \subseteq N_i(SO(2))$. Clearly, this implies $|\Delta(C(X))| = |\Delta(L(X))|$ for all $X \subseteq N_i(SO(2))$ and all i. Macaulay posets satisfying this equality are called *shadow symmetric*.

Theorem 20 (Engel, Leck [31]). *Let P be a Macaulay poset. If P is shadow symmetric, then P additive.*

According to the above theorem, $SO(2)$ and its dual are additive.

Theorem 21 (Engel, Leck [31]). *The subword order $SO(2)$ is shadow increasing and weakly shadow increasing.*

Unfortunately, the dual of $SO(2)$ is obviously not shadow increasing In fact, this poset is even *shadow decreasing* (see [31] for a proof). However, for some applications (see section 5.1) the weak shadow increase property can serve as a substitute.

Let us now briefly discuss the case of larger alphabets. In [14] a Kruskal-Katona type theorem for $SO(n)$ with $n \geq 2$ was presented but there is a mistake in the proof, as pointed out by Danh and Daykin [26]. They also provided an example showing that the statement itself is not true at all for $n > 2$.

Daykin [28] introduced the *V-order*, an extension of the VIP-order for $SO(n)$ with $n \geq 2$. He conjectured that this order is a Macaulay order for $SO(n)$. For $n \geq 3$, a counterexample to this conjecture is given in [44]. Even worse, this example and a tedious case study yield the following result.

Theorem 22 (Leck [44]). *If $n > 2$, then the subword order $SO(n)$ is not a Macaulay poset.*

4.5 The linear lattice

The linear lattice L^n is another example of a poset which is not representable as a cartesian product of other posets. This poset is defined to be the collection of all proper nonempty subspaces of $\mathrm{PG}(n,2)$ ordered by inclusion.

Note that $2^{n+1}-1$ points of $\mathrm{PG}(n,2)$ are just $(n+1)$-dimensional non-zero binary vectors $(\beta_1,\ldots,\beta_{n+1})$. Using the lexicographic ordering of the points, let us represent each subspace $a \in L^n$ by its characteristic vector, i.e. by the $(2^{n+1}-1)$-dimensional binary vector $(\alpha_{2^{n+1}-1},\ldots,\alpha_1)$, where α_i corresponds to the i^{th} point of $\mathrm{PG}(n,2)$.

For two subspaces $a, b \in L^n$, we say that a is greater than b in the order $\mathcal{O}$ if the characteristic vector of a is greater than the one of b in the lexicographic order. Now for $t > 0$ and $A \subseteq L^n_t$ denote

$$\hat{\Delta}(A) = \{x \in L^n_0 \mid x \leq y,\ y \in A\}$$

and consider the SMP for the levels L^n_t and L^n_0.

Theorem 23 (Bezrukov, Blokhuis [13]). *Let $n \geq 1$ and $t > 0$. Then any IS of the order $\mathcal{O}_t$ has minimal shadow $\hat{\Delta}(\cdot)$. The shadow $\hat{\Delta}(\cdot)$ of any IS is an IS itself.*

However, as it is shown in [13], this poset is not Macaulay for $n \geq 3$.

5 Extremal ideals in Macaulay posets

In this section we will be concerned with some optimization problems for which solutions are known for a rich class of Macaulay posets.

Let P be a poset, and let $I\!R^+$ denote the set of nonnegative real numbers. Furthermore, let there be a *weight function* $w : P \mapsto I\!R^+$ on P. If $w(x) = w(y)$ whenever $r(x) = r(y)$, the function $w(\cdot)$ is called *rank-symmetric*. If $w(\cdot)$ is a rank-symmetric weight function and $w(x) \leq w(y)$ whenever $r(x) < r(y)$, then $w(\cdot)$ is called *monotone*. Now define the weight of a subset $X \subseteq P$ as $w(X) = \sum_{x \in X} w(x)$.

5.1 Generated ideals of minimum weight

Consider the problem of constructing an antichain $X \subseteq P$ of given cardinality $m \leq d(P)$ such that the ideal generated by X has minimum weight for some monotone weight function.

This problem was considered by Frankl [33] for the Boolean lattice. For chain products, the problem was solved by Clements [19] who generalized preliminary results of Kleitman [38] and Daykin [27]. A further generalization is due to Engel [30] who provided a solution for the class of Macaulay posets P such that P and P^* are graded, additive, and shadow increasing. Unfortunately, the subword order $SO(2)$ is not included in this class since its dual is not shadow increasing (see section 4.4). Therefore, Engel and Leck [31] gave the following strengthening which applies to the classical Macaulay posets as well as to $SO(2)$.

Theorem 24 (Engel, Leck [31]). *Let P be a Macaulay poset such that P and P^* are weakly shadow increasing. Furthermore, let $m \leq d(P)$ be a positive integer, and put $i := \min\{j \mid m \leq |P_j|\}$ and $a := \min\{b \mid b + |P_{i-1}| - |\Delta(C(b, P_i))| = m\}$. Then the set*

$$X := C(a, P_i) \cup (P_{i-1} \setminus \Delta(C(a, P_i)))$$

is an antichain of size m. Moreover, $w(I(X)) \leq w(I(Y))$ holds for all antichains $Y \subseteq P$ with $|Y| = m$ with respect to any monotone weight function.

This theorem provides a sufficient condition for a poset to be Sperner (cf. [31] for details).

Corollary 3 *Let P be a Macaulay poset such that P is not an antichain. If P and P^* are weakly shadow increasing, then P is graded and has the Sperner property, i.e. the size of maximum antichain of P is equal to $\max_i |P_i|$.*

5.2 Ideals with maximum number of maximal elements

Now consider a dual to the last problem. Namely, we are looking now for an ideal of a given size, which has maximum number of maximal elements. In order to present a solution to this problem, we first introduce quasispheres. A *quasisphere* of size m in a ranked poset P is a set of the form

$$P_0 \cup P_1 \cup \cdots \cup P_i \cup C(a, P_{i+1}),$$

where the numbers a and i are (uniquely) defined by $m = \sum_{j=0}^{i} |P_j| + a$, $0 \leq a < |P_{i+1}|$. Obviously, any quasisphere is an ideal.

Theorem 25 (Engel, Leck [31]). *Let P be a Macaulay poset such that P and P^* are weakly shadow increasing. Then a quasisphere of size m has the maximum number of maximal elements in the class of all ideals of size m in P.*

Clearly, the set of maximal elements of some ideal is an antichain. For Boolean lattices, a related problem was considered by Labahn [40]. He determined the maximum size of an antichains X such that the ideal generated by X contains exactly m elements of P_i.

5.3 Maximum weight ideals

Now consider a problem of finding an ideal $I^* \subseteq P$ such that $w(I) \geq w(I)$ for any other ideal $I \subseteq P$ with $|I| = |I^*|$. We call this problem the Maximum Weight Ideal problem (MWI for brevity). Denote $w_i = w(x)$ for any $x \in P_i$.

The MWI problem is closely related to the edge-isoperimetric problems (cf. Section 3.2 and [8, 11] for more details) and was first considered by Bernstein and Steiglitz in [5] for the Boolean lattice and applied to a problem in coding theory.

Theorem 26 (Bernstein, Steiglitz [5]). *If $\preceq$ is a lexicographic order, then for any $m = 0, \ldots, 2^n$ the set $C(m, B^n)$ is a solution to the MWI problem for B^n with respect to any monotone weight function.*

Clements and Lindström in [23] extended Theorem 26 to the chain products in the case $w_i = i$ for all i, where a similar solution with respect to the lexicographic order was obtained by using Theorem 4. It turns out that the MWI problem is a direct consequence of the shadow minimization problem, as presented in the following theorem (see [6, 30]).

Theorem 27 *Let $(P, \leq, \preceq)$ be a rank-greedy Macaulay structure with a monotone weight function. Then the set $C(m, P)$ is a solution to the MWI problem for P.*

What if the weight function is not monotone? It is easily seen that if $w_0 \geq w_1 \geq \cdots \geq w_n$ then a solution to the MWI problem is attained on a quasisphere for any ranked poset P. For some less trivial nonmonotone weight functions a solution to the MWI is known for the Boolean lattice.

Theorem 28 (Ahlswede, Katona [4]). *Consider the Boolean lattice and let $\preceq$ be the lexicographic order.*

a. If $w_0 \leq w_1 \leq \cdots \leq w_{i-1} \geq w_i \geq \cdots \geq w_n$, then a solution to the MWI problem is attained on an intersection of $C(m', B^n)$ with a quasisphere for some $m' \leq m$.

b. If $w_0 \geq w_1 \geq \cdots \geq w_{i-1} \leq w_i \leq \cdots \leq w_n$, then a solution to the MWI problem is attained on an union of $C(m', B^n)$ with a quasisphere for some $m' \leq m$.

Bezrukov and Voronin in [17] proposed a new approach to this problem which significantly explores the Macaulayness property. They showed that similar result holds for the chain products. Note that the methods of neither [4] nor [17] provide exact values of m'. The corresponding results describe the situation just qualitatively and only ensure that such m' does exist. We guess that the approach of [17] can be extended to qualitatively describe maximum weight ideals for any rank-symmetric weight function, at least for the Boolean lattice and the products of chains.

Let us return back to Theorem 27. Evidently, the MWI and the SMP are closely related. The principal question is what should we claim on the solutions to the MWI problem in order to deduce the Macaulayness of the corresponding poset? Counterexamples show that the nestedness in the MRI problem on a poset P does not imply the Macaulayness of P in general. Thus, the SM problem is, in a sense, a more difficult problem than MWI.

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