# Extremal Ideals of the Lattice of Multisets

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**Abstract**

We present here constructions of ideals $A$ of the poset of $n$-vectors $(x_1, ..., x_n)$ with integer entries, ordered coordinatewise, on which the maximal and minimal values of $W_\varphi(A) = \sum_{x \in A} \varphi(\sum_{i=1}^n x_i)$ are achieved for a given unimodal function $\varphi$. As a consequence we get a new approach to prove the well-known Clements-Lindström Theorem [6].

## 1 Introduction

Let $d_1, ..., d_n$ be arbitrary integers and $d_1 \leq d_2 \leq ... \leq d_n$.

**Definition 1** *Denote by $I$ the collection of all $n$-dimensional vectors $(x_1, ..., x_n)$ with integer coordinates from the range $0 \leq x_i \leq d_i$, for $1 \leq i \leq n$. We call $I$ by a grid and the number $n$ by dimension of $I$.*

**Definition 2** *The coordinate sum of a vector $x = (x_1, ..., x_n) \in I$ is called the weight of $x$ and denoted by $\|x\|$. If $\varphi(z)$ is a positive-valued real function defined for nonnegative $z$ then we call the number $W_\varphi(x) = \varphi(\|x\|)$ by the modified weight of $x \in I$ with respect to the function $\varphi$, and the number $W_\varphi(A) = \sum_{x \in A} W_\varphi(x)$ by the modified weight of a set $A \subseteq I$ with respect to $\varphi$.*

**Definition 3** *A subset $A \subseteq I$ is called ideal, if for any $x = (x_1, ..., x_n) \in A$ and $y = (y_1, ...y_n)$ with $y_i \leq x_i$ for $1 \leq i \leq n$ it follows $y \in A$.*

For a lot of applications (see [1,2,3], for example) one has to find among all the $m$-element subsets $A \subseteq I$ a subset, on which an extremal value of some function $f$ is achieved. Usually it is sufficient to proceed the search of an extremal subset $A$ in the class of ideals only. Moreover, very often it is possible to choose a function $\varphi$ such that the value of the function $f$ equals the modified weight of the ideal with respect to $\varphi$. A typical example of such situation is the problem of finding an $m$-element subset of $I$ with maximal possible number of induced Hamming edges, i.e. the pairs $(x, y)$, $x, y \in A$, with $\rho(x, y) = 1$, where $\rho$ is the Hamming metric [2,3,4,5]. After some simple arguments which allow to restrict the class of considered subsets by ideals, it is not difficult to show that the number of Hamming edges for an ideal $A$ equals $W_\varphi(A)$ with $\varphi(t) = t$.

A important aspect in the theory of extremal subsets is the analysis of proof techniques. Such analysis, as a rule, allows not only to understand the structure one works with more deeply, but also to simplify the known proofs, and so to design more powerful methods. It is also always of interest to prove that some problem is a consequence of some other problem. For example, it is known that the mentioned problem of finding an $m$-element subset with maximal size of $\{(x, y)\}$ with $\rho(x, y) = 1$ is equivalent to the problem of finding an $m$-element ideal with maximal possible weight [4,6]. We will show below that these two problems are equivalent to the well-known Theorem of Clements and Lindström [6] (see also Corollary 1) and, in fact, present a new simple proof of it.

The main problem we are concerned in this paper with, is to find an ideal $A \in I$ of a fixed size on which an extremal value of $W_\varphi(A)$ is achieved. Such type of function is the general type of a symmetric function, i.e. which depends only of the *number* of vectors of weight $i$ in $A$ for all $i$.

Denote $w_i = \varphi(\|x\|)$ for $\|x\| = i$ and $d = d_1 + ... + d_n$. If $d_1 = ... = d_n = 1$, i.e. when $I$ is the $n$-dimensional unit cube, then the extremal ideals were constructed in [1] in the case when $\varphi$ is of one of the following types:

$$w_0 \leq ... \leq w_i \geq w_{i+1} \geq ... \geq w_d \quad \text{and}$$

$$w_0 \geq ... \geq w_i \leq w_{i+1} \leq ... \leq w_d.$$

In our paper we prove similar results for arbitrary numbers $d_1, ..., d_n$.

The paper is organized by the following. The next two sections are devoted to the solution of our problem for some special class of subsets. The main result of these sections is Theorem 1, which is formulated in Section 2. Section 3 consists of the proof of Theorem 1 and in Section 4 with the help of this Theorem we obtain the solutions of our problem for the class of arbitrary ideals of $I$.

## 2 Some auxiliary propositions and the approach

**Definition 4** *We say that $x = (x_1, ..., x_n) \in I$ is greater $y = (y_1, ..., y_n) \in I$ in the lexicographic order (denotation $x \succ y$), if either $x_1 > y_1$ or if $x_i = y_i$ for $i = 1, 2, ..., s$ and some $s < n$, and $x_{s+1} > y_{s+1}$.*

Denote by $L^n(m)$ the collection of the first $m$ vectors of $I$ in the lexicographic order.

**Definition 5** *We say that a set $A \subseteq I$ is an initial segment if $A = L^n(m)$ for some $m$.*

Denote

$$I^t(i) = \{x \in I \,:\, x_i = t\}.$$

**Definition 6** *A set $A \subseteq I$ is called i-compressed if $A \cap I^t(i)$ is an initial segment of $I^t(i)$ for all $t = 0, ..., d_i$.*

**Lemma 1** *Let $n \geq 3$, $A \subseteq I$ and $A$ is i-compressed for $i = 1, ..., n$. Then either*

*(i)* $A = L^n(|A|)$, *or*

*(ii) there exist numbers $a, p$ and $q$ for which*

$$A = L^n(m') \cup \{(x_1, ..., x_n) \in I \, : \, x_1 = a+1, \; x_2 = ... = x_{n-1} = 0, \quad 0 \leq x_n \leq q < p \leq d_n\},$$

*where* $m' = a \cdot \prod_{i=2}^{n}(d_i + 1) + (p+1) \cdot \prod_{i=2}^{n-1}(d_i + 1)$.

*Proof.*
The proof of the Lemma is based on the fact that for such $A \subseteq I$ the condition $x = (x_1, ..., x_n) \in A$ implies $y = (y_1, ..., y_n) \in A$ for any $y \prec x$, maybe with some exceptions defined in $(ii)$.

Assume that there exists an index $i$ for which $x_i = y_i = t$. Then $x, y \in I^t(i)$ and $y \in A$, since $A$ is $i$-compressed.

Let now $x_i \neq y_i$ for $i = 1, ..., n$. Obviously $x_1 > y_1$. If $x_n > y_n$, then consider the vector

$$z = (x_1, ..., x_{n-1}, y_n).$$

It is easy to verify that $x \succ z \succ y$ and $x_1 = z_1$, $y_n = z_n$. Therefore, $z \in A$ since $A$ is 1-compressed, and hence $y \in A$ since $A$ is $n$-compressed.

If $x_n < y_n$ and there exists an index $i$, $2 \leq i \leq n-1$, such that $y_i < d_i$, we consider the vector

$$z = (y_1, ..., y_{i-1}, y_i + 1, 0, ..., 0, x_n).$$

One has $x \succ z \succ y$, and similarly $z \in A$ and $y \in A$.

If $x_n < y_n$, $y_i = d_i$ for $2 \leq i \leq n-1$ and $x_1 > y_1 + 1$, then considering the vector

$$z = (x_1 - 1, d_2, ..., d_{n-1}, x_n),$$

for which $x \succ z \succ y$, one also gets $y \in A$.

Let now $x_1 = y_1 + 1$, $x_n < y_n$, $y_i = d_i$ for $2 \leq i \leq n-1$ and there exists an index $j$, $2 \leq j \leq n-1$, for which $x_j \neq 0$. Then, considering the vector

$$z = (x_1, ..., x_{j-1}, x_j - 1, x_{j+1}, ..., y_n),$$

we get again $x \succ z \succ y$ and so, $y \in A$.

We have to consider the last case:

$$x = (a+1, 0, ..., 0, x_n), \quad y = (a, d_2, ..., d_{n-1}, y_n)$$

with $x_n < y_n$. In this case it is impossible to guarantee $y \in A$. However, if $y \in A$, then $(i)$ holds. If $y \notin A$ then $A$ is the union of the initial segment $S$ with lexicographically greatest vector of the form

$$(a, d_2, ..., d_{n-1}, p),$$

with $p = y_n - 1$, and $q = x_n + 1$ vectors of the form $(a+1, 0, ..., 0, t)$ with $0 \leq t \leq q < p \leq d_n$. One has only notice that the size of $S$ equals

$$a \cdot \prod_{i=2}^{n}(d_i + 1) + (p+1) \cdot \prod_{i=2}^{n-1}(d_i + 1).$$

□

**Remark 1** *For $n = 1, 2$ the Lemma is not true in general.*

Denote

$$\begin{aligned} I_k &= \{x \in I : \|x\| = k\} \\ I_{k,k-1} &= I_k \bigcup I_{k-1} \\ A_k &= A \cap I_k. \end{aligned}$$

Up to the end of this Section we assume $A \subseteq I_{k,k-1}$.

**Definition 7** *A set $A$ is called monotone if $A$ is an intersection of $I_{k,k-1}$ with some ideal of $I$.*

Let

$$F(A) = w_{k-1} \cdot |A_{k-1}| + w_k \cdot |A_k|,$$

where $w_k$, $w_{k-1}$ are some nonnegative numbers and denote by $L^n_{k,k-1}(m)$ the $m$-element subset, which is the intersection of $I_{k,k-1}$ and some initial segment of $I$.

**Theorem 1**

*$(i)$ if $w_k > w_{k-1}$ then the maximal value of $F(A)$ among all $m$-element monotone subsets of $I_{k,k-1}$ is achieved on $L^n_{k,k-1}(m)$;*

*$(ii)$ if $w_k < w_{k-1}$ then the maximal value of $F(A)$ among all $m$-element monotone subsets of $I_{k,k-1}$ is achieved on arbitrary collection of $m$ vectors of $I_{k-1}$ for $m \leq |I_{k-1}|$ and on the union of $I_{k-1}$ with arbitrary collection of $m - |I_{k-1}|$ vectors of $I_k$ for $m > |I_{k-1}|$;*

*$(iii)$ if $w_k = w_{k-1}$ then $F(A) = const$ for any $A \subseteq I_{k,k-1}$, $|A| = m$.*

Since the propositions $(ii)$ and $(iii)$ of Theorem 1 obviously hold, we will prove $(i)$ only.

Let $A \subseteq I_{k,k-1}$ be a monotone set and $|A| = m$.

**Definition 8** *The set $A$ is called optimal if $F(A) \geq F(B)$ for any $m$-element monotone subset $B \subseteq I_{k,k-1}$.*

Denote

$$\begin{aligned} P(A) &= \{x = (x_1, ..., x_n) \in I : \|x\| > k \text{ and } y_i \leq x_i,\ 1 \leq i \leq n, \text{ for some } y \in A\}, \\ T(A) &= \{x = (x_1, ..., x_n) \in I : |x| < k-1 \text{ and } x_i \leq y_i,\ 1 \leq i \leq n, \text{ for some } y \in A\}. \end{aligned}$$

**Definition 9** *The set $[A] = A \cup P(A) \cup T(A)$ is called the closure of $A$.*

We will prove in the next Lemma that among all optimal subsets there exists a subset such that for any $i, j$

$$A \cap I^i_{k,k-1}(j) = L^n_{k-i,k-i-1}(m_{i,j}),$$

where $m_{i,j} = |A \cap I^n_{k,k-1}(j)|$. Furthermore, in Lemma 3 we will show that the closure of $L^n_{k,k-1}(m)$ is an initial segment of $I$. These facts will allow later to propose that among all monotone subsets $A$ satisfying Lemma 2, there exists a subset $A$, such that $[A]$ satisfies Lemma 1. It will give us a possibility to determine the structure of $A$ very precisely and to prove Theorem 1 in Section 3. Notice that the proof of Theorem 1 could be simplified by using the Clements-Lindström theorem [6]. However, we did not use this theorem because, as it is shown in Section 4, it is an easy consequence of Theorem 1.

In order to formulate our next propositions it is convenient to introduce the operators of compression $C(A)$ and $C_j(A)$. For a subset $A \subseteq I_{k,k-1}$ put

$$\begin{aligned} C(A) &= L^n_{k,k-1}(|A|) \quad \text{and} \\ C_j(A) &= \bigcup_{i=0}^{d_j} C(A^i(j)), \end{aligned}$$

where $A^i(j) = A \cap I^i(j)$ and the operator $C$ in the right hand side is applied in $n-1$ dimensions.

**Lemma 2** *Let $A \subseteq I_{k,k-1}$ be a monotone set and Theorem 1 is true in $n-1$ dimensions. Then there exists a subset $A' \subseteq I_{k,k-1}$ such that $|A| = |A'|$, $F(A) \geq F(A')$ and $C_j(A') = A'$ for $1 \leq j \leq n$.*

*Proof.*
Let us fix the index $j$ and replace each subset $A_k \cap I^i(j)$ with the equal-sized initial segment of $I^i_k(j)$. Proceed by the same way with $A_{k-1} \cap I^i(j)$ for all $i$. We obtain a subset $B \subseteq I_{k,k-1}$ for which $F(A) = F(B)$ holds. However $B$ may be nonmonotone. It may happen only if for some $i$, $0 \leq i \leq d_j$ the set $B^i(j)$ is nonmonotone in $I^i(j)$. Replacing now this $B^i(j)$ with $L^n_{k-i,k-i-1}(|B^i(j)|)$ in the corresponding $(n-1)$-subgrids, we obtain a monotone set $D$. Since Theorem 1 is true for $I^i(j)$, then $F(D) \geq F(B)$. Now if $C_j(D) = D$, then jump to the last paragraph of the proof. Otherwise there is an index $i$ for which

$$D^i(j) \neq L^{n-1}_{k-i,k-i-1}(|D^i(j)|).$$

Denote by $u$ the lexicographically greatest vector of $D^i_{k-1}(j)$, and by $v$ the lexicographically least vector of $I^i_k(j) \setminus D^i_k(j)$. Then $u \succ v$.

Now we are going to show that if $C_j(D) \neq D$ then there exists a monotone subset $E$ such that $F(E) \geq F(D)$ and the sum of the lexicographical numbers of vectors of $D$ is greater then one for $E$. Indeed, if $E' = (D \setminus u) \bigcup v$ is monotone then let $E = E'$ and we are done. If $E'$ is nonmonotone then consider three cases:

**Case 1.** Let $i > 0$, $|D^i_k(j)| = |D^{i-1}_{k-1}(j)|$ and either $i = d_j$ or $|D^i_{k-1}(j)| > |D^{i+1}_k(j)|$ for $i < d_j$. Denote by $r$ the vector obtained from $v$ by decreasing $v_i$ on 1 and denote

$$E = (D \setminus u) \cup r.$$

**Case 2.** Let $i < d_j$, $|D^i_{k-1}(j)| = |D^{i+1}_k(j)|$ and either $i = 0$ or $|D^{i-1}_{k-1}(j)| > |D^i_k(j)|$ for $i > 0$. Denote by $s$ the vector obtained from $u$ by increasing $u_i$ on 1 and let

$$E = (D \setminus s) \cup v.$$

**Case 3.** Let $0 < i < d_i$ and $|D^{i-1}_{k-1}(j)| = |D^i_k(j)|$, $|D^i_{k-1}(j)| = |D^{i+1}_k(j)|$. Denote

$$E = (D \setminus \{u, s\}) \cup \{r, v\}.$$

In any case the set $E$ is monotone and $F(E) \geq F(D)$. If now $C_j(E) \neq E$ then we can repeat the procedure above. Since the sum of the lexicographic numbers of vectors of the new sets cannot decrease infinitely, then after a finite number of transformations we obtain a monotone subset $G$ such that $C_j(G) = G$.

To complete the proof of the whole Lemma we have to repeat our procedure for $j = 1, 2, ...$ until we obtain the desired set $A'$. □

**Lemma 3** *If $A \subseteq I_{k,k-1}$ and $C(A) = A$ then $[A]$ is an initial segment in $I$.*

*Proof.*
Denote by $u$ the lexicographically greatest vector of $[A]$ and by $v$ the lexicographically least vector of $I \setminus [A]$. If $v \succ u$ then the Lemma is true. Assume $v \prec u$.

Let $\|v\| \leq \|u\|$. There exists an index $j$ such that $u_i = v_i$ for $1 \leq i \leq j-1$ and $v_j < u_j$. Consider an arbitrary vector $r \in I$ such that $\|r\| = \|v\|$, $r_i = u_i$ for $1 \leq i \leq j-1$, $r_j > v_j$ and $r_i \leq u_i$ for $j+1 \leq i \leq n$. Such vector $r$ already exists since $\|u\| \geq \|v\|$. By the definition of $T(A)$, $r \in [A]$. Furthermore, $v \prec r \prec u$. Since $[A] \cap I_t$ is a collection of the first vectors of $I_t$ in the lexicographic order for any $t$ (see [6], Lemma 3), then $v \in A$. A contradiction. Therefore, $\|v\| \geq \|u\|$.

If $\|v\| > k$, then by the definition of $P(A)$ there exists a vector $s \in I$ such that $s \notin [A]$, $\|s\| = \|v\| - 1$ and $s_i \leq v_i$ for $1 \leq i \leq n$, i.e. $s \prec v$, which contradicts to the choice of $v$. Therefore $\|v\| = k$ and hence, $\|u\| = k-1$. However, $v \prec u$ contradicts to $C(A) = A$. □

## 3 Proof of Theorem 1

We proceed by induction on $n$. Since we use Lemma 1, which is true for $n > 3$ only, we have to prove Theorem 1 for $n = 2$.

Let $I$ be a $d_1 \times d_2$ ($d_1 \le d_2$) two-dimensional grid. Notice that for $d_1 \le 2$ the Theorem is true. Assume that it is true for all two-dimensional $d \times d_2$ grids with $d \le d_1$ and consider the case $d = d_1$. Let $A \subseteq I_{k,k-1}$ be a monotone subset.

Without loss of generality we may assume $A^0(1) \neq \emptyset$, i.e. $1 \le |A^0(1)| \le 2$. Consider $(d_1 - 1) \times d_2$ subgrid $I'$, obtained from $I$ by deleting the column $I^0(1)$ and denote $A' = A \setminus A^0(1)$. Then $A'$ is a monotone subset of $I_{k-1,k-2}$ and

$$F(A) = w_1 \cdot |A_k \cap I^0(1)| + w_2 \cdot |A_{k-1} \cap I^0(1)| + F(A'),$$

where $F(A')$ is computed in the subgrid $I'$. Replacing $A'$ with $L^2_{k-1,k-2}(|A'|)$ in the subgrid $I'$, we obtain a set $B \subseteq I$. Using the inductive hypothesis for $B$, one gets $F(B) \ge F(A)$. Now if $|B^0(1)| = 2$, then $B = L^2_{k,k-1}(|B|)$. If $|B^0(1)| = 1$, then denote by $r$ the vector $(0, k) \in I \setminus B$ and by $u$ and $v$ the lexicographically greatest and least vectors of $B_k$ and $B_{k-1}$ respectively. It is easy to verify that either $(B \setminus u) \cup r$ or $(B \setminus v) \cup r$ is equal to $D = L^2_{k,k-1}(|B|)$ and $F(D) \ge F(B)$. Therefore the Theorem is true for $n = 2$.

Let $n \ge 3$ and $A \subseteq I_{k,k-1}$ be a monotone set. By Lemma 2, there exists a monotone set $B \subseteq I_{k,k-1}$, such that $C_j(B) = B$ for $1 \le j \le n$. Denote by $m_j$ the maximal value of the $j$-th entry of vectors of $B$. If there exists a vector $u \in I_k \setminus B$ such that $u_j < m_j$ and each vector obtained from $u$ by decreasing of any nonzero entry (i.e. compatible with u) belongs to $B_{k-1}$, then replace the lexicographically greatest vector $v$ of $B_k \cap I^{m_j}(j)$ with $u$, and the set $D = B^{m_j}(j)$ with the set $L^{n-1}_{k-m_j,k-m_j-1}(|D|)$ in the subgrid $I^{m_j}(j)$. Since the sum of the lexicographical numbers of vectors cannot increase infinitely, then after a sufficiently large number of the transformations we obtain a monotone subset $E$ such that $F(E) \ge F(B)$ and $C_j(E) = E$ for $1 \le j \le n$. Furthermore, if there exists a vertex $u \in I_k \setminus E$, such that any compatible with it vertex of $I_{k-1}$ belongs to $E$ then $u_j = j$ for $1 \le j \le n$. Now we will show that for $[E]$ one can apply Lemma 1.

First notice that

$$(P(E))^i(j) \subseteq P(E^i(j)).$$

Moreover, equality holds here. Indeed, assume that there exists a vector

$$u \in P(E^i(j)) \setminus P(E^i(j)).$$

Denote by $v$ the vector obtained from $u$ by replacing $u_j = i$ with $i - 1$. Then $\|v\| \ge k$. Without loss of generality we may let $v \notin P(E^{i-1}(j))$. Since $E$ is monotone set then for all $t \ge 1$ and $i \ge 1$ one has

$$|[E] \cap I^i_{t-1}(j)| \le |[E'] \cap I^{i-1}_t(j)|,$$

which leads to a contradiction when $\|v\| > k$. If $\|v\| = k$ then since $E$ is monotone then each vector of weight $k - 1$ compatible with $v$, belongs to $E$ and since $v \notin E$ then by definition of $E$, $v_j = m_j$, i.e. $i = m_j + 1$ and $u \in E^i(j) = \emptyset$, which is a contradiction too.

Further, notice that

$$(T(E))^i(j) \supseteq T(E^i(j)).$$

We show by reversed induction on $j$, that $[E]^i(j)$ is an initial segment in $I^i(j)$. By Lemma 3 it is so if $i = m_j$. Consider the case $i = m_j - 1$ and assume that $(T(E))^i(j) \neq T(E^i(j))$. Denote by $G \subseteq I^i_{k-1,k-2}(j)$ the set, for which $G_{k-1} = E^i_{k-1}(j)$ and $G_{k-2}$ is obtained from $E^{i+1}_{k-1}(j)$ by decreasing on 1 the $j$-th entry in all vectors $u \in E^{i+1}_{k-1}(j)$. By the definition of $E$, $[E]^i(j) = [G]$ and the set $G$ is an initial segment of $I^i(j)$. Therefore by Lemma 3, $[E]^i(j)$ is an initial segment of the subgrid $I^i(j)$. For $i < m_j - 1$ the inductive step may be covered similarly.

Therefore we can apply Lemma 1 to $E$. If the case $(i)$ of it holds then the Theorem 1 is obviously true. Let the case $(ii)$ holds and $E \neq L^n_{k,k-1}(m)$ (here $m = |E|$). Then by Lemma 1 either $E = N \cup \{u\}$ or $E = N \cup \{u, v\}$, where $N = L^n_{k,k-1}(m')$ and $u, v$ are of the form

$$u = (a+1, 0, ..., 0, p), \qquad v = (a+1, 0, ..., 0, p+1)$$

with $\|u\| = k-1$, $\|v\| = k$.

On the other hand by Lemma 1 either $N = J \setminus \{s\}$ or $N = J \setminus \{s, r\}$, where

$$J = I^0_{k,k-1}(1) \cup ... \cup I^a_{k-a,k-a-1}(1)$$

and $s, r$ ($\|r\| = k-1$, $\|s\| = k$) are the two lexicographically greatest vectors of $J$. Now if $E = N \cup u$ then $N = J \setminus s$, since $E \neq L^n_{k,k-1}(m)$. Therefore $G = (E \setminus u) \cup s = L^n_{k,k-1}(m)$ and $F(G) > F(E)$. If $E = N \cup \{u, v\}$, then if $N = J \setminus s$, then denote $G = (E \setminus v) \cup s$ and if $N = J \setminus \{r, s\}$, then denote $G = (E \setminus \{u, v\}) \cup \{r, s\}$. In the both cases $G = L^n_{k,k-1}(m)$ and $F(G) \geq F(E)$. $\square$

Using the similar techniques one can prove the proposition on the minimization of $F$, which is obtained from Theorem 1 by replacing the inequalities $w_k > w_{k-1}$ with $w_k < w_{k-1}$, and the words "maximum" with "minimum".

Let $A \subseteq I_k$. Denote

$$T_i(A) = T(A) \cap I_{k-i}, \quad P_i(A) = P(A) \cap I_{k+i}.$$

**Corollary 1** *(The Clements-Lindström theorem [6]).*

$(i)$ $|T_1(L^n_k(m))| \leq |T_1(A)|$ *for any* $A \subseteq I_k$, $\quad |A| = m$;

$(ii)$ $|P_1(L^n_k(m))| \geq |P_1(A)|$ *for any* $A \subseteq I_k$, $\quad |A| = m$.

*Proof.*
Consider an arbitrary set $B = A \cup T_1(A)$ and let $|B| = t$. Denote $D = L^n_{k,k-1}(t)$. By Theorem 1(i), $F(D) \geq F(B)$. Since $|D| = t$ and $D$ is a monotone subset then

$$|D_k| \geq |A|, \quad |D_{k-1}| \leq |T_1(A)|, \quad L^n_k(m) \subseteq D_k.$$

Hence, $T_1(L^n_k(m)) \subseteq D_{k-1}$, i.e. $|T_1(L^n_k(m))| \leq |T_1(A)|$. The proposition $(ii)$ may be proved similarly. $\square$

Let $k > l$. Denote

$$I_{k,l} = I_k \cup I_{k-1} \cup ... \cup I_l,$$

and let $L^n_{k,l}(m)$ denotes the $m$-element set which is an intersection of $I_{k,l}$ and an initial segment of $I$. For $A \subseteq I_{k,l}$ consider the function

$$F_{k,l} = w_k \cdot |A_k| + w_{k-1} \cdot |A_{k-1}| + ... + w_l \cdot |A_l|,$$

where $w_i$ are some fixed nonnegative numbers.

**Definition 10** *The set $A \subseteq I_{k,l}$ is called $(k,l)$-monotone if for any $i$, $l+1 \le i \le k$, the inclusion $A_{i-1} \supseteq T_1(A_i)$ holds.*

**Definition 11** *We call the set $A$ quasisphere if*

$$A = I_0 \cup I_1 \cup ... \cup I_{t-1} \cup A'$$

*for some $t$, where $A'$ is the collection of the first $|A'|$ vectors of $I_t$ in the lexicographic order.*

**Corollary 2** *Let $A$ be a $(k,l)$-monotone set and $|A| = m$. Then*

*(i) if $w_l \le ... \le w_k$ ($w_l \ge ... \ge w_k$), then the maximum (minimum) of $F_{k,l}$ among all $m$-element subsets of $I_{k,l}$ is achieved on $L^n_{k,l}(m)$;*

*(ii) if $w_l \ge ... \ge w_k$ ($w_l \le ... \le w_k$), then the maximum (minimum) of $F_{k,l}$ among all $m$-element subsets of $I_{k,l}$ is achieved on the intersection of a quasisphere with $I_{k,l}$.*

*Proof.*
It is sufficient to prove $(i)$ for $w_l \le ... \le w_k$. Replace $A_i$ to $L^n_i(|A_i|)$ with $i = l, ..., k$. We obtain a set $B$ which is $(k,l)$-monotone by Corollary 1. Furthermore, for each pair $I_i$, $I_{i-1}$ we replace $B_{i,i-1}$ with $L^n_{i,i-1}(|B_{i,i-1}|)$. It is easy to show that all the new sets are $(k,l)$-monotone and by Theorem 1 every nontrivial transformation leads to increasing of $F_{k,l}$. Hence, after a finite number of steps we obtain a set $D$, which is invariant under such transformation. Denote by $u$ the lexicographically greatest vector of $D$ and by $v$ the lexicographically least vector of $I_{k,l} \setminus D$. If $v \succ u$ then $D = L^n_{k,l}(m)$. So let $v \prec u$, $\|u\| = p$, $\|v\| = q$.

If $q > p$ then any vector $r$ such that $\|r\| = q - 1$ and $r_i \le v_i$, $1 \le i \le n$, belongs to $A$ and the set $(D \cap I_{p+1,p}) \setminus u$ is monotone in $I_{p+1,p}$. Hence, $E = (D \setminus u) \cup v$ is $(k,l)$-monotone and $F_{k,l}(E) > F_{k,l}(D)$. Therefore either we can transform $D$ into $L^n_{k,l}(m)$, or $q < p$.

If $q < p$, then consider the following sequence of vectors $r_1, r_2, ..., r_p$. The vectors $r_1, ..., r_{u_n}$ are obtained from $u$ by decreasing $u_n$ to 1, 2,..., $u$ respectively. The next $u_{n-1}$ vectors are obtained from $r_{u_n}$ by decreasing it's $(n-1)$-st entry to $1, 2, ..., u_{n-1}$ and so forth with the $(n-2)$-nd entry, $(n-3)$-rd,.... It is easy to verify that $\|r_i\| = \|u\| - i$ and that the vector $r$ of weight $q$ from this sequence is the lexicographically greatest in $T_{p-q}(\{u\})$ and $r \in D$. Hence, if $v \prec r$ then $v \in D$ by the definition of $D$. If $v \succ r$, then $v \succ u$ and we get a contradiction. $\square$

## 4 The extremal ideals

**Theorem 2**

*(i) if* $w_0 \leq w_1 \leq ... \leq w_{i-1} \geq w_i \geq ... \geq w_d$ *for some* $i$*, then the maximum of* $W_\varphi$ *is achieved on the intersection of some initial segment with a quasisphere .*

*(ii) if* $w_0 \geq w_1 \geq ... \geq w_{i-1} \leq w_i \leq ... \leq w_d$ *for some* $i$*, then the maximum of* $W_\varphi$ *is achieved on the union of some initial segment with a quasisphere.*

*Proof.*
Notice that for any $k, l$ and $m$.

$$L^n_{k,l}(m) \cup P_1(L^n_{k,l}(m)) = L^n_{k+1,l}(m') \text{ and}$$

$$L^n_{k,l}(m) \cup T_1(L^n_{k,l}(m)) = L^n_{k,l-1}(m''),$$

where $m'$ and $m''$ are the cardinalities of the unions. If $A$ is an ideal then after replacing $A_k$ with $L^n_k(|A_k|)$ for all $k$, the obtained set is an ideal either by the Clements-Lindström Theorem and the value of $W_\varphi$ on it equals to $W_\varphi(A)$. So we may assume $A_k = L^n_k(|A_k|)$.

$(i)$ Replace $A_{i-1,0}$ with $L^n_{i-1,0}(|A_{i-1,0}|)$. The obtained set $B$ is an ideal by the Clements-Lindström Theorem and $W_\varphi(A) \leq W_\varphi(B)$ by the Corollary 1. Replace $B_{i+1}$ with $P_1(B_i)$, $B_{i+2}$ with $P_1(B_{i+1})$ ,..., $B_{i+t}$ with $P_1(B_{i+t-1})$ for $t$ such that $B_{i+t+1} \subseteq P_1(B_{i+t})$ and $B_s \neq \emptyset$ for $s > i+t+1$. We obtain a set $D$ and

$$D_{i+t,0} = L^n_{i+t,0}(|D_{i+t,0}|).$$

Hence, $D$ is the intersection of some initial segment in $I$ with a quasisphere.

$(ii)$ Replace $A_{d,i}$ with $L^n_{d,i}(|A_{d,i}|)$, where $d = d_1 + ... + d_n$. By the Corollary 2, for the obtained ideal $B$ we have $W_\varphi(A) \leq W_\varphi(B)$. Replace now $B_{i-1}$ with $T_1(B_i)$, $B_{i-2}$ with $T_1(B_{i-1})$,...,$B_{i-t+1}$ with $T_1(B_{i-t+2})$, $B_{i+t} \supseteq T_1(B_{i-t+1})$ with $I_s$ for $s \leq i-t-1$. Then for the obtained set $D$ we have $D_{d,i-t+1} = L^n_{d,i-t+1}(|D_{d,i-t+1}|)$ and $D_{i-t,0}$ is a quasisphere. Hence, $D$ is the union of a quasisphere and an initial segment. □

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