On the Construction of Solutions of a Discrete Isoperimetric Problem in Hamming Space

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Abstract

This article is a study of the solution set of a discrete isoperimetric problem.

1 Introduction and list of results

Denote by B^n the *n*-dimensional unit cube, i.e. the collection of all *n*-dimensional vectors with each coordinate equal to 0 or 1, and by $\rho(\alpha, \beta)$ the Hamming distance between vertices in B^n . Let $A \subseteq B^n$. Then $\tilde{\alpha} \in A$ is called an inner point of A iff $S_1^n(\tilde{\alpha}) \subseteq A$, and a boundary point otherwise (where $S_r^n(\tilde{\alpha})$ is the ball around a point $\tilde{\alpha}$ with the radius rin the metric ρ). Denote by P(A) ($\Gamma(A)$) the collection of all inner (boundary) points of A. A set $A \subseteq B^n$ is said to be *optimal* if $|\Gamma(A)| \leq |\Gamma(B)|$ for any $B \subseteq B^n$, |B| = |A|.

This article is a study of the collection of optimal subsets of B^n . It is known (see [1,3,6]) that one of the *m*-element optimal sets is the standard arrangement L_m^n , defined as the initial segment of length *m* of the following numbering (denoted below by \mathcal{L}) of the vertices of B^n . The vector $\tilde{0} = (0, ..., 0)$ is given the number 1. Suppose that all the vectors in B^n with coordinate sum (norm) less than *k* have already been numbered, along with some vectors of norm *k*, We give the next largest natural number to the lexicographically greatest unnumbered vertex of B^n of norm *k*.

Let

$$S(A) = \{ \tilde{\alpha} \in A : S_1^n(A) \cap P(A) = \emptyset \}$$

be the collection of free points of A. A set A is said to be *critical* if $S(A) = \emptyset$. The empty set will be regarded as a critical set too. Denote by \mathcal{N}_m^n the collection of all *m*-element optimal noncritical subsets of B^n . It is shown in Section 2 that the construction of the sets in this class reduces to the construction of the optimal critical sets.

It is known (see [1,3,4]) that if L_m^n is the critical set then any *m*-element optimal subset is also critical. Such numbers *m* are called critical cardinalities. Denote by \mathcal{M}_m^n the collection of all optimal subsets with the critical cardinality *m*. A description of all the sets in the class \mathcal{M}_m^n is given in [4]. In is easy to show (see Corollary 3.7.1) that the number of critical cardinalities equals 2^{n-1} .

Denote by \mathcal{K}_m^n the collection of all *m*-element optimal critical subsets of B^n . Notice that there exist some *m*, for which $\mathcal{K}_m^n = \emptyset$. One of our main results is Theorem 3.1,

in which it is shown that if A is an optimal set, then so is P(A). On the basis of this Theorem we obtain in Section 4 a sufficient condition for $\mathcal{K}_m^n = \emptyset$. Thus, all the solutions of the isoperimetric problem are found for such cardinalities m. In the same Section we also show that the number of these cardinalities asymptotically equals 2^{n-3} . Moreover, a necessary and sufficient condition for $\mathcal{K}_m^n = \emptyset$ is found.

In Section 5 we show that for any subset of B^n there exists an optimal subset of a cube of higher dimension that in a certain sense is analogous to it in structure. This result reflects well the difficulties arising in the description of all solutions of the isoperimetric problem and at the same time gives a method for constructing certain optimal critical subsets.

2 A theorem on reducibility

Let *m* be an arbitrary integer with $1 < m < 2^n$. We show how to construct all optimal *m*-element sets. Denote by m_0 the critical cardinality closest to *m* from below, i.e., $m_0 = m - |S(L_m^n)|$, and let $m_1, ..., m_l$ ($m_o < m_1 < \cdots < m_l \le m$) be all the integers for which there exist optimal critical sets of those cardinalities in B^n . We construct l + 1 families F_i of *m*-element sets. For this we consider all m_i -element optimal critical sets ($0 \le i \le l + 1$) and for each such set *A* we construct $\binom{2^{n-m_i}}{m-m_i}$ sets by adding $m - m_i$ arbitrary points in $B^n \setminus A$ to *A*. Thus, we get a family F_i .

Theorem 2.1 The sets in $\bigcup_{i=0}^{l} F_i$ are pairwise distinct and optimal. If A is an m-element optimal set, then $A \in F_i$ for some *i*.

Proof.

Let A be an optimal set. If $A \in \mathcal{K}_m^n$, then $A \in F_l$. Let $A \in \mathcal{N}_m^n$ and $\tilde{\alpha} \in S(A)$. We show that $A \setminus \tilde{\alpha}$ is an optimal set. Assume that $A \setminus \tilde{\alpha}$ is nonoptimal set and let $B \subseteq B^n$ be an optimal (m-1)-element set. Then for $\tilde{\beta} \in B^n \setminus B$ one has

$$|P(B \cup \hat{\beta})| \ge |P(B)| > |P(A \setminus \tilde{\alpha})| = |P(A)|,$$

which contradicts the optimality of A. Since $P(A \setminus \tilde{\alpha}) = P(A)$, then $S(A \setminus \tilde{\alpha}) = S(A) \setminus \tilde{\alpha}$, i.e. the removal of a free point $\tilde{\alpha}$ from A leads neither to the appearance of a new free point in $A \setminus \tilde{\alpha}$ nor to the disapiarance of an old free point.

Thus, by removing free points from A sufficiently many times, we get that for any $A \subseteq \mathcal{N}_m^n$ there exists a critical optimal set B, uniquely determined by A of cardinality |A| - |S(A)|. We now prove that $|B| \ge m_0$, i.e. $|B| = m_i$ for some i. For this it suffices to show that $|S(A)| \le |S(L_m^n)|$. Assume that $s_1 = |S(A)| > |S(L_m^n)| = s_2$. We construct an optimal subset C by removing s_2 arbitrary free points from A. Since $L_{m-s_2}^n$ is critical, $|P(A)| = |P(C)| = |P(C \setminus \tilde{\beta})| > |P(L_{m-s_2-1}^n)|$ for $\tilde{\beta} \in S(C)$. Contradiction. Thus, $|B| \ge m_0$.

Note that if A is an optimal q-element set and $m_0 \leq q \leq m$, then $|P(A)| = |P(L_{m_0}^n)|$. Therefore, if B is an optimal m_i -element set, then by adding $m - m_i$ arbitrary points to B we get an optimal set. Since for any $A \in \mathcal{N}_m^n$ there exists precisely one optimal critical set $B \subseteq A$, $|B| = m - |S(A)| \geq m_0$, the sets from $\bigcup_{i=0}^l F_i$ are pairwise distinct.

3 General properties of optimal subsets

Let $1 \leq m \leq 2^n$. Represent it in the form $m = \sum_{i=0}^k \binom{n}{i} + \delta$, $0 \leq \delta < \binom{n}{k+1}$. Notice that k and δ are uniquely determined by m. It is known [1, Theorem 6.1], that if A is an optimal m-element subset, then there exists a point $\tilde{\alpha} \in A$, such that $S_k^n(\tilde{\alpha}) \subseteq A$. Obviously $|S_{k+1}^n(\tilde{\alpha})| \geq |A|$. Therefore, each optimal subset A contains a ball of maximal radius.

Theorem 3.1 If A if an optimal set then so is P(A).

In order to prove Theorem 3.1 we need some auxiliary propositions.

Lemma 3.1 $P(L_n^m)$ is an optimal set.

Since $P(L_n^m)$ is a standard set, the proof follows.

We partition the cube B^n with respect to the coordinate x_i and denote by $A^0(i)$ and $A^1(i)$ the parts of a set A in the resulting (n-1)-dimensional subcubes $x_i = 0$ and $x_i = 1$ respectively. Let $p(m, n) = |P(L_n^m)|$. The coordinate x_i is said to be admissible for a set A if

 $p(|A^{0}(i)|, n-1) \le |A^{1}(i)|$ and $|A^{0}(i)| \ge |A^{1}(i)|,$

or if

 $p(|A^1(i)|, n-1) \le |A^0(i)|$ and $|A^0(i)| \le |A^1(i)|$.

Denote by $\pi_{i_1,...,i_t}(A)$ the projecting operator consisting of replacement of 0 by 1 and 1 by 0 in the coordinates with indices $i_1, ..., i_t$ in all vectors in A. The following two assertions are proved in [1] (respectively, as see Lemma 4.1a and Lemma 4.2 slightly modified).

Lemma 3.2 Let A be an optimal set and let x_i be it's admissible coordinate. Then $A^0(i)$ and $A^1(i)$ are optimal subsets in (n-1) dimensions, $P(A) = P(A^0(i)) \cup P(A^1(i))$ and the following holds for $|A^0(i)| \ge |A^1(i)|$:

a) If $p(|A^{0}(i)|, n-1) < |A^{1}(i)|$, then $\pi_{i}(A^{0}(i)) \supseteq P(A^{1}(i) \text{ and } \pi_{i}(A^{1}(i)) \supseteq P(A^{0}(i))$. b) If $p(|A^{0}(i)|, n-1) = |A^{1}(i)|$, then $\pi_{i}(A^{1}(i)) = P(A^{0}(i))$.

Lemma 3.3 For an optimal set A the following holds

a) If $p(|A^{0}(i)|, n-1) = |A^{1}(i)|$ or $p(|A^{1}(i)|, n-1) = |A^{0}(i)|$ for any i = 1, ..., n, then either $A = B^{n}$, or $A = S_{r}^{n}(\tilde{\alpha})$ for some $\tilde{\alpha}$, and $r \leq n-2$. b) If $p(|A^{0}(i)|, n-1) > |A^{1}(i)|$ or $p(|A^{1}(i)|, n-1) = |A^{0}(i)|$, then $A = S_{n-1}^{n}(\tilde{\alpha})$.

Denote $P_i(A) = P(P_{i-1}(A)), P_0(A) = A.$

Lemma 3.4 Let A be an optimal subset and let x_i be it's admissible coordinate. Then $P_2(A) = P_2(A^0(i)) \cup P_2(A^1(i)).$

Proof.

Obviously, $P_2(A) \subseteq P_2(A^0(i)) \cup P_2(A^1(i))$. We show that the inverse inclusion also holds. Let $\tilde{\alpha} \in P_2(A^0(i))$, i.e. $S_2^{n-1}(\tilde{\alpha}) \subseteq A^0(i)$. Then $S_1^{n-1}(\tilde{\alpha}) \subseteq P(A^0(i))$ and $\pi_i(S_1^{n-1}(\tilde{\alpha})) \subseteq A^1(i)$. Consequently, $S_2^n(\tilde{\alpha}) = S_2^{n-1}(\tilde{\alpha}) \cup S_2^{n-1}(\pi_i(\tilde{\alpha})) \subseteq A$, i.e. $\tilde{\alpha} \in P_2(A)$. The proof of the Lemma in the case $\tilde{\alpha} \in P_2(A^1(i))$ is quite similar.

Denote by $\varphi_i(A)$ the following transformation of a set $A \subseteq B^n$, called compression. We partition the cube B^n with respect to the coordinate x_i , and replace the sets $A^0(i)$ and $A^1(i)$ by the standard arrangements of the same cardinalities in the corresponding subcubes. The proof of the two following assertions is given in [1] (see lemmas 3.1 and 2.2 respectively).

Lemma 3.5 If A is an optimal subset, so is $\varphi_i(A)$.

We say that a set A is *i*-compressed if $\varphi_i(A) = A$.

Lemma 3.6 Let $A \subseteq B^n$ be *i*-compressed for i = 1, ..., n and |A| = m. Then:

a) Either $A = L_m^n$ or $m = 2^{n-1}$ and $A = L_{m+1}^n \setminus \tilde{\alpha}_m$, where $\tilde{\alpha}_m$ is the m-th point in order \mathcal{L} .

b) If A is an optimal set then $A = L_m^n$ for n > 2.

It follows from Lemmas 3.2 and 3.5 that if x_i is an admissible coordinate for A, then it is admissible also for $\varphi_i(A)$.

Proof of Theorem 3.1. We use induction on n. For n = 2, 3 the Theorem is obviously true, so let us proceed the inductive step. Let $A \subseteq B^n$ is an optimal set and |A| = m. Below we describe a serie of transformations carrying A into L_m^n and prove the equality $|P_2(A)| = |P_2(L_m^n)|$, which yields the Theorem when Lemma 3.1 and the equality $|P(A)| = |P(L_m^n)|$ are taken in account.

Construct the set $B = \varphi_i(A)$. Considering Lemmas 3.1, 3.4, 3.5 and the inductive hypothesis, we have that

$$|P_2(A^0(i))| = |P_2(B^0(i))|, \qquad |P_2(A^1(i))| = |P_2(B^1(i))|$$

and

$$|P_2(B)| = |P_2(B^0(i))| + |P_2(B^1(i))| = |P_2(A)|.$$

Let us repeat this procedure for an admissible coordinate of a set B, then for an admissible coordinate of the newly obtained set, and so on. Denote by l(A) the sum of numbers of $\tilde{\alpha} \in A$ in order \mathcal{L} . Note that if $\varphi_i(A) \neq A$ then $l(\varphi_i(A)) < l(A)$. Therefore, after a finite number of such compressions we obtain a set D, which is *i*-compressed for $i \in \{i_1, ..., i_s\}$, where $x_{i_1}, ..., x_{i_s}$ is the collection of all admissible coordinates of the set D. It will be assumed that among all such sets the minimum of the functional l is attained on D. We note that |D| = |A| and $|P_2(D)| = |P_2(A)|$.

If s = 0 or s = n then the Theorem follows from Lemmas 3.3b and 3.6b. So let 0 < s < n. Without loss of generality we assume that $i_1 = 1, ..., i_s = s$. We show that $D = L_m^n$.

Let us partition B^n with respect to the coordinates $x_1, ..., x_s$ into subcubes of dimension (n - s). If for at least one of the resulting parts of the partition of D tat is in an

n - s-dimensional subcube there exists a coordinate (from among the coordinates of the corresponding subcube) satisfying the condition of Lemma 3.2a, then it would be admissible for the set D. Consequently, for each part of D the condition of Lemma 3.3 holds (in n - s dimensions). According to this lemma, each such part is a ball. It is not hard to show, that the centers of these balls (we call them *basic balls*) are in the origins of coordinates of the corresponding (n - s)-subcubes. Note that the radius of at least one basic ball equals n - s - 1, since otherwise any of the coordinates x_i , $i \ge s + 1$, would be admissible for the set D.

Denote by $\tilde{\alpha}$ the greatest point of D in order \mathcal{L} and by $\tilde{\beta}$ the least point of $B^n \setminus D$ in order \mathcal{L} . Considering the foregoing,

$$\tilde{\alpha} = (\alpha_1, ..., \alpha_s, 0, ..., 0, \underbrace{1, ..., 1}_p)$$

for some p. If $\tilde{\beta}$ is greater $\tilde{\alpha}$ in \mathcal{L} , then $D = L_m^n$. So let $\tilde{\alpha}$ is greater $\tilde{\beta}$ and let us show $\tilde{\beta} \in D$. If $\alpha_i = \beta_i$ for some $i \leq s$, then $\tilde{\beta} \in D$, since D is *i*-compressed. Therefore, it is sufficient to consider the case

$$\beta = (\bar{\alpha}_1, \dots, \bar{\alpha}_s, \beta_{s+1}, \dots, \beta_n),$$

where $\bar{\alpha}_i$ denotes the logical negation of the entry α_i . Moreover, since $\tilde{\beta}$ belongs to one of the basic balls, then $\tilde{\beta} \in D$ iff $\tilde{\gamma} \in D$, where

$$\tilde{\gamma} = \left(\bar{\alpha}_1, ..., \bar{\alpha}_s, 0, ..., 0, \underbrace{1, ..., 1}_q\right)$$

and $q = \sum_{j=s+1}^{n} \beta_j$. So, without loss of generality we assume that $\tilde{\beta} = \tilde{\gamma}$, and D contains all the vectors that are greater than

$$(\bar{\alpha}_1, \dots, \bar{\alpha}_s, \underbrace{1, \dots, 1}_p, 0, \dots, 0)$$

in order \mathcal{L} . Moreover, it is sufficient to consider the case when there is no vector $\tilde{\epsilon} \in B^n$, such that $\tilde{\alpha} < \tilde{\epsilon} < \tilde{\beta}$ and the vector pairs $(\tilde{\alpha}, \tilde{\epsilon})$ and $(\tilde{\epsilon}, \tilde{\beta})$ have at least one common coordinate from among the first *s* coordinates. This case is possible only if $\tilde{\beta}$ is the immediate predecessor of the vector

$$\tilde{\alpha}' = (\alpha_1, ..., \alpha_s, \underbrace{1, ..., 1}_p, 0, ..., 0)$$

in order \mathcal{L} . We will show below that $\tilde{\alpha}$ and $\tilde{\beta}$ must then have the form

$$\tilde{\alpha} = (\underbrace{0, 1, ..., 1}_{s}, 0, ..., 0, \underbrace{1, ..., 1}_{p}), \quad \tilde{\beta} = (1, 0, ..., 0, 0, ..., 0, \underbrace{1, ..., 1}_{p+s-2}),$$

or

$$\tilde{\alpha} = (\underbrace{1,...,1}_{s}, 0,...,0,\underbrace{1,...,1}_{p}), \quad \tilde{\beta} = (\underbrace{0,...,0}_{s}, 0,...,0,\underbrace{1,...,1}_{p+s-1})$$

Case 1. Let $\alpha_1 + \alpha_2 + \cdots + \alpha_s < s$, i.e. there exists an index j, for which $\alpha_j = 0$. Let $j \leq s$ be the maximal such index. If j > 1, then $\alpha = \beta_1$, and so $\alpha_1 = \beta_1$. A contradiction. Therefore, $\alpha_1 = 0$ and $\alpha_2 = \alpha_3 = \cdots = \alpha_s = 1$. Further, if s = 1 then x_1 is not admissible coordinate for the set D when p = n - 1. If $0 , then <math>D^0(1)$ (it is not a ball) must have an admissible coordinate by Lemma 3.3, i.e. $s \geq 2$.

If $\|\tilde{\alpha}\| = t > \|\beta\|$, then

$$(\underbrace{0,0,1,...,1}_{s},0,...,0,\underbrace{1,...,1}_{p})\in D,$$

and hence $\tilde{\beta} \in D$, i.e. $\tilde{\beta}$ is not the immediate predecessor of $\tilde{\alpha}'$ in order \mathcal{L} . Therefore, $\|\tilde{\beta}\| = t$. Note that $t \leq n - s + 1$. If $t \leq n - s - 1$, then there does not exist a basic ball of radius n - s - 1. Consequently, either t = n - s + 1 or t = n - s.

a) Let t = n - s + 1. In this case vectors $\tilde{\alpha}, \tilde{\beta}$ have the form

$$\tilde{\alpha} = (\underbrace{0, 1, ..., 1}_{s}, 0, ..., 0, \underbrace{1, ..., 1}_{p}) \quad \hat{\beta} = (\underbrace{1, 0, ..., 0}_{s}, 1, ..., 1).$$

Then $|P(D \setminus \tilde{\alpha})| = |P(D)| - p$ and $|P(D \setminus \tilde{\beta})| = |P(D)| + n - s$. Consequently, if p < n - s (i.e. $s \ge 2$), then $|P((D \setminus \tilde{\alpha}) \cup \tilde{\beta})| > |P(D)|$, i.e. D is nonoptimal subset. If s = 2, then x_1 is not admissible for D. So, $\tilde{\beta} \in D$.

b) Let t = n - s. Denote

$$B_{1} = \{ \tilde{\gamma} \in D : \tilde{\gamma} \text{ has the form } (1, 0, 0, ..., 0, \gamma_{s+1}, ..., \gamma_{n}) \}$$

$$B_{2} = \{ \tilde{\gamma} \in D : \tilde{\gamma} \text{ has the form } (0, 1, 0, ..., 0, \gamma_{s+1}, ..., \gamma_{n}) \}.$$

Invert the first and the second entries in all vectors of B_1 and B_2 . This gives a set E, for which |P(E)| = |P(D)| but l(E) < l(D). Contradiction.

Case 2. Let $\alpha_1 + \alpha_2 + \ldots + \alpha_s = s$, i.e.

$$\tilde{\alpha} = (\underbrace{1,...,1}_{s},0,...,0,\underbrace{1,...,1}_{p}) \quad \tilde{\beta} = (0,...,0,\underbrace{1,...,1}_{q}).$$

If $\|\tilde{\beta}\| < \|\tilde{\alpha}\| - 1$ then $\tilde{\beta}$ is not the immediate predecessor of $\tilde{\alpha}'$ in order \mathcal{L} . Therefore, $\|\tilde{\beta}\| = \|\tilde{\alpha}\| - 1 = t$. In a way similar to that in Case 1 we get that either t = n - s or t = n - s - 1.

a) Let t = n - s. Then $|P(D \setminus \tilde{\alpha})| = |P(D)| - p$ and $|P(D \cup \beta)| = |P(D)| + n - s$. If p < n - s (i.e. $s \ge 1$), then $|P((D \setminus \tilde{\alpha}) \cup \tilde{\beta})| > |P(D)|$, which contradicts the optimally of D. But if s = 1 then x_1 is not admissible coordinate for D. Hence, $\tilde{\beta}$ must be in D.

b). Let t = n - s - 1. Since $\beta \in D$, for s > 1 there does not exist a basic ball of radius n - s - 1, i.e. D must have another admissible coordinate (for s < n). But if s = 1, then the vectors $\tilde{\alpha}$ and $\tilde{\beta}$ have the form $\tilde{\alpha} = (1, 0, 1, ..., 1)$, $\tilde{\beta} = (0, 0, 1, ..., 1)$. Therefore, D is a ball of radius n - 2 centered in (1, 0, ..., 0) and the minimum of the functional l is not attained on D.

Therefore in all cases we get that $\tilde{\beta} \in D$, i.e., D is a standard arrangement.

Corollary 3.1.1 If A is an optimal set, then so is $P_s(A)$ for s = 1, ..., n.

Corollary 3.1.2 Let $A \subseteq B^n$ be an optimal set and |A| = m. Then $|P_s(A)| = |P_s(L_m^n)|$ for s = 1, ..., n.

Corollary 3.1.2, in particular, implies that if A is an optimal *m*-element set and the number m is represented in the form $m = \sum_{i=0}^{k} \binom{n}{i} + \delta$, $0 \leq \delta < \binom{n}{k+1}$, then $|P_k(A)| = |P_k(L_m^n)|$. Moreover it is obvious that $P_{k+1}(A) = P_{k+1}(L_m^n) = \emptyset$. Thus, Corollary 3.1.2. not only implies the equality $P_k(A) \neq \emptyset$, which is equivalent to $S_k^n(\tilde{\alpha}) \in A$ for some $\tilde{\alpha}$ (cf. [1], Theorem 6.1), but also yields the possibility of determining the number of such points. This number equals to $|P_k(L_m^n)|$ and can be found from known formulas [7].

Corollary 3.1.3 Let A be an optimal critical set and $|A| = m < \binom{n}{2} + n + 1$. Then A is a union of exactly $|P(L_m^n)|$ balls of radius 1.

This corollary can be regarded as a description of the structure of optimal sets of small cardinality.

For the solution of many applied problems (see [1,5] for example) it is required to consider the set

$$G_t(A) = \{ \tilde{\alpha} \in B^n \setminus A : \rho(\tilde{\alpha}, A) \le t \}$$

and to find a subset $A \subseteq B^n$ with |A| = m on which the minimum of the functional $|G_t(A)|$ is attained. Denote this problem by I_t .

Corollary 3.1.4 If A is a solution of problem I_t , then it is also a solution of problem I_r for any r > t.

The Corollary is proved by applying Theorem 3.1 to the set $B^n \setminus A$ with the equality $G_t(A) = (B^n \setminus A) \setminus P_t(B^n \setminus A)$. This equality in particular implies, that L_m^n is a solution of problem I_t .

Denote by J_t the problem of constructing a set A of fixed cardinality with $\|\tilde{\alpha}\| = k$ for any $\tilde{\alpha} \in A$ on which the functional $|G_t(A) \cap \{\tilde{\alpha} : \|\tilde{\alpha}\| = k + t\}|$ attains a minimum.

Corollary 3.1.5 If A is a solution of problem J_t , then it is also a solution of problem J_r for any r > t.

This corollary follows from Corollary 3.1.4. and from the fact that L_m^n is a solution of problem I_t .

Let R is an arbitrary numbering of vertices of B^n by the numbers $1, 2, ..., 2^n$. We say that a set A is generated by the numbering R if A consists of the vertices of B^n with indices 1, 2, ..., m. It is of definite interests to find all numberings generating optimal subsets for any m. Such numberings are called *optimal numberings*. There are many examples of extremal problems on B^n with solutions generated by appropriate numberings. The possibilities of representation of optimal sets by the optimal numberings are reflected in the following theorem.

Theorem 3.2 Let $A \subseteq B^n$ be an optimal *m*-element critical set generated by an optimal numbering R. Then *m* is a critical cardinality.

Proof.

Let $\tilde{\alpha} \in A$ has the greatest number in R. Then $A \setminus \tilde{\alpha}$ is an optimal set and $|P(A)| > |P(A \setminus \tilde{\alpha})|$. Thus

$$|P(A)| = |P(L_m^n)|, \quad |P(A \setminus \tilde{\alpha})| = |P(L_{m-1}^n)|, \quad |P(L_m^n)| > |P(L_{m+1}^n)|.$$

and the proof follows.

For that follows we need a criterion enabling us to determine whether a number m is a critical cardinality. Denote by $\tilde{\alpha} = (\alpha_1, ..., \alpha_n)$ the greatest point of L_m^n in the order \mathcal{L} .

Lemma 3.7 The number m is a critical cardinality iff $\alpha_n = 1$.

Proof.

Assume that m is a critical cardinality and $m = \sum_{i=0}^{k} \binom{n}{i} + \delta$, $0 \leq \delta < \binom{n}{k+1}$. We write $C = P(L_m^n) \cap B_k^n$ and let $\tilde{\beta} = (\beta_1, ..., \beta_n)$ be the greatest point in C in order \mathcal{L} . Since $P(L_m^n) = L_{m'}^n$, where $m' = \sum_{i=0}^{k-1} \binom{n}{i} + c$, c = |C|, then $\tilde{\beta} \in \Gamma(L_m^n \setminus \tilde{\alpha})$. However, it is possible only if $\beta_n = 0$ and $\rho(\tilde{\alpha}, \tilde{\beta}) = 1$, i.e. $\alpha_n = 1$. The proof of sufficiency is similar.

Corollary 3.7.1 The number of critical cardinalities of the cube B^n equals 2^{n-1} .

4 On the existence of sets in the class \mathcal{K}_m^n

It is not hard to show that for some *m* there may not exist an optimal critical *m*-element set. For example, let $m = \sum_{i=0}^{k} {n \choose i} + 1$. By Corollary 3.1.2, any optimal set of such cardinality must consist of a ball of radius *k* and another point $\tilde{\alpha}$ outside the ball. The point $\tilde{\alpha}$ may be chosen arbitrarily; but obviously $\tilde{\alpha} \in S(A)$ always. Thus, $\mathcal{K}_m^n = \emptyset$ in this case. In the next theorem we prove a sufficient condition under which $\mathcal{K}_m^n = \emptyset$. As before, let $p(m, n) = |L_m^n|$.

Theorem 4.1 If p(m,n) is a critical cardinality, then $\mathcal{K}_m^n = \emptyset$.

In order to prove this theorem we need some auxiliary propositions. Let

$$G(A) = \{ \tilde{\alpha} \in B^n : \rho(\tilde{\alpha}, A) \le 1 \},\$$

a neighborhood of A. Note that $A \subseteq G(A)$.

Lemma 4.1 If A is an optimal set of critical cardinality. Then G(A) is also an optimal set of critical cardinality.

Proof.

We prove this by induction on n. For n = 2 it can be checked directly. Let us proceed the inductive step for $n \ge 3$.

We say that $A \subseteq B^n$ is a set of standard arrangement type if it can be obtained from L^n_m by some permutation of the coordinates of B^n and a "translation" of the resulting

set by some vector $\tilde{\gamma} \in B^n$. We call a coordinate x_i of B^n minimal for a set A if $||A^0(i) - |A^1(i)|| \leq ||A^0(j) - |A^1(j)||$ for j = 1, ..., n.

Without loss of generality we may assume that A is not a set of standard arrangement type, since in opposite case the Lemma is obviously true. For an optimal set A of a critical cardinality $m = \sum_{i=0}^{k} {n \choose i} + \delta$ the following assertions hold (see [1] Lemma 6.1, Theorem 6.2 and Corollary 6.2 respectively):

1. If x_i is a minimal coordinate for A then $A^0(i)$ and $A^1(i)$ are optimal critical subsets of critical cardinalities (in n-1 dimensions) and the cardinality of at least one of them equals $\sum_{i=0}^{k} {n-1 \choose i}$.

2. There exists a point $\tilde{\alpha} \in A$ such that $S_k^n(\tilde{\alpha}) \subseteq A \subseteq S_{k+2}^n(\tilde{\alpha})$.

3. If $S_k^n(\tilde{\alpha}) \subseteq A \subseteq S_{k+1}^n(\tilde{\alpha})$, then A is a set of standard arrangement type.

Without loss of generality we assume that $S_k^n(\tilde{0}) \subseteq A \subseteq S_{k+2}^n(\tilde{0})$, where $\tilde{0} = (0, ..., 0)$. Let x_i be a minimal coordinate for A. Since A is not a ball, then it f follows from Lemmas 3.2 and 3.3 that

$$P(A^0(i)) \subseteq \pi_i(A^1(i)), \quad P(A^1(i)) \subseteq \pi_i(A^0(i)),$$

which implies $G(A) = G(A^0(i) \cup G(A^1(i)))$, where the operator G in the right hand side is applied in n-1 dimensions. Denote B = G(A).

Case 1. Assume that $\delta \leq \binom{n-1}{k}$. a) Let $|A^0(i)| \geq |A^1(i)|$. By Assertion 1,

$$|A^{0}(i)| = \sum_{i=0}^{k} \binom{n-1}{i}, \quad |A^{1}(i)| = \sum_{i=0}^{k-1} \binom{n-1}{i} + \delta.$$

By Assertion 2, $A^0(i) = S_k^{n-1}(\tilde{0})$ and $S_{k-1}^{n-1}(\tilde{0}) \subseteq A^1(i) \subseteq S_{k+1}^{n-1}(\tilde{0})$. Therefore, taking into account the inductive hypothesis,

$$B^{0}(i) = S_{k+1}^{n-1}(\tilde{0}), \quad S_{k}^{n-1}(\tilde{0}) \subseteq B^{1}(i) \subseteq S_{k+2}^{n-1}(\tilde{0}),$$

and $B^1(i)$ is an optimal set of critical cardinality. Consider a set $C \subseteq B^n$ obtained from B by replacing $B^1(i)$ by the standard arrangement of the same cardinality in the subcube $x_i = 1$. We have that

$$P(B^0(i)) \subseteq \pi_i(B^1(i))$$
 and $P(B^1(i)) \subseteq \pi_i(B^0(i)),$

hence

$$P(C^{0}(i)) = P(B^{0}(i)) \subseteq \pi_{i}(C^{1}(i))$$
 and $P(C^{1}(i)) \subseteq \pi_{i}(C^{0}(i)).$

Consequently,

$$P(B) = P(B^{0}(i)) \cup P(B^{1}(i))$$
 and $P(C) = P(C^{0}(i)) \cup P(C^{1}(i))$

Since $B^1(i)$ is an optimal set (in n-1 dimensions), then |P(B)| = |P(C)|. Note, that C is a set of standard arrangement type. Therefore, B is an optimal set. Since $|B^1(i)|$ is a critical cardinality and $B^0(i)$ is a ball, then |B| is a critical cardinality either.

b) Let $|A^{0}(i)| < |A^{1}(i)|$. In this case

$$|A^{1}(i)| = \sum_{i=0}^{k} \binom{n-1}{i},$$

and since $S_k^n(\tilde{0}) \subseteq A$, then

$$|A^{0}(i)| \ge \sum_{i=0}^{k} \binom{n-1}{i} = |A^{1}(i)|.$$

A contradiction.

Case 2. Assume $\delta > \binom{n-1}{k}$. a) Let $|A^0(i)| \ge |A^1(i)|$. Then $S_k^{n-1}(\tilde{0}) \subseteq A^0(i) \subseteq S_{k+2}^{n-1}(\tilde{0})$ and $A^1(i) = S_k^{n-1}(\tilde{\alpha})$

for some $\tilde{\alpha} = (\alpha_1, ..., \alpha_n)$, such that $\alpha_n = 1$. It is not hart to show that $1 \leq ||\tilde{\alpha}|| \leq 2$, since otherwise the inclusion $S_k^n(\tilde{0}) \subseteq A \subseteq S_{k+2}^n(\tilde{0})$ is violated.

If $\|\tilde{\alpha}\| = 1$, i.e. α_i is the only nonzero entry of $\tilde{\alpha}$, then since $P(A^0(i)) \subseteq \pi_i(A^1(i))$ and $A^0(i)$ is an optimal set by Assertion 1 (in n-1 dimensions), then

$$P(A^{0}(i)) \cap \{(\gamma_{1}, ..., \gamma_{n}) \in B^{n} : \gamma_{i} = 0\& \|\tilde{\gamma}\| = k+1\} = \emptyset.$$

Consequently,

$$A^{0}(i) \cap \{(\gamma_{1}, ..., \gamma_{n}) \in B^{n} : \gamma_{i} = 0\& \|\tilde{\gamma}\| = k + 2\} = \emptyset,$$

i.e., $S_k^n(\tilde{0}) \subseteq A \subseteq S_{k+1}^n(\tilde{0})$. By Assertion 3, A is a set of standard arrangement type and the Lemma holds.

If $\|\tilde{\alpha}\| = 2$, then since $P(A^0(i)) \subseteq \pi_i(A^1(i))$ and $P(A^1(i)) \subseteq \pi_i(A^0(i))$, we get, that the same holds for the set B and so B is an optimal set of critical cardinality by the inductive hypothesis. The rest of the proof is completely analogous to Case 1a) and is thus omitted.

b) Let $|A^0(i)| < |A^1(i)|$. Then

$$S_k^{n-1}(\tilde{0}) \subseteq A^1(i) \subseteq S_{k+2}^{n-1}(\tilde{0}) \text{ and } A^0(i) = S_k^{n-1}(\tilde{0}).$$

Note that

$$|A^{1}(i)| = \sum_{i=0}^{k} \binom{n-1}{i} + \left(\delta - \binom{n-1}{k}\right)$$

and $A^i(i)$ is an optimal set of critical cardinality. By Assertion 2, there exists a point $\tilde{\alpha} = (\alpha_1, ..., \alpha_n)$ with $\alpha_i = 1$, such that $S_k^{n-1}(\tilde{\alpha}) \subseteq A^1(i) \subseteq S_{k+2}^{n-1}(\tilde{\alpha})$. It is not hard to show, that $1 \leq \|\tilde{\alpha}\| \leq 2$. Therefore, $S_k^n(\tilde{\alpha}) \subseteq A \subseteq S_{k+2}^n(\tilde{\alpha})$ and the proof of the Lemma is reduced to the Case 2a) by applying the arguments presented there to the set A "translated" by the vector $\tilde{\alpha}$.

Proof of Theorem 4.1. Assume the contrary, i.e. $A \in \mathcal{K}_m^n \neq \emptyset$. Since A is a critical set, then $A = P(A) \cap G(P(A))$. Taking into account that P(A) is an optimal set by Theorem

3.1 and applying Lemma 4.1 to the set P(A), we get that A is an optimal set of critical cardinality, i.e., $A \notin \mathcal{K}_m^n$.

A number m is called a singular cardinality, if for any m', $m_0 < m' \leq m$, there does not exists an optimal m'-element critical subset of B^n . In this case all the optimal critical m-element subsets belong to the family F_0 (see Section 2), i.e. their structure is known [4].

Corollary 4.1.1 If p(m,n) is a critical cardinality, then m is a singular cardinality.

The proof follows from the equalities $p(m_0, n) = p(m_0 + 1, n) = \cdots = p(m, n)$.

Denote by X(n) the number of singular cardinalities determined by Corollary 4.1.1.

Corollary 4.1.2 $X(n) \simeq 2^{n-3}$ as $n \mapsto \infty$.

Proof.

Let $m = \sum_{i=0}^{k} {n \choose i} + \delta$ be a number satisfying the condition of Theorem 4.1, and let m_0 be the critical cardinality closest to m from below. Consider the set $L_{m_0}^n$ and denote by $\tilde{\alpha}$ the greatest vector of $L_{m_0}^n$ in order \mathcal{L} . By Lemma 3.7, $\alpha_n = 1$. Moreover, $k \leq \|\tilde{\alpha}\| \leq k+1$.

Suppose first that $\|\tilde{\alpha}\| = k + 1$. We show that $\alpha_{n-1} = 1$. Assume that $\alpha_{n-1} = 1$ and consider the vector $\tilde{\beta} = (\alpha_1, ..., \alpha_{n-2}, 0, 0)$. Obviously, $\tilde{\beta} \in P(L_{m_0}^n)$ and $\tilde{\beta}$ is the greatest vector of $P(L_{m_0}^n)$ in order \mathcal{L} . But since $\beta_n = 0$ and $p(m, n) = p(m_0, n)$ is a critical cardinality, we have a contradiction.

On the other hand, if $\alpha_{n-1} = \alpha_n = 1$, then the vectors $\delta = (\alpha_1, ..., \alpha_{n-2}, 1, 0)$ and $\tilde{\gamma} = (\alpha_1, ..., \alpha_{n-2}, 0, 1)$ are in $P(L_{m_0}^n)$, and if $\tilde{\delta} > \tilde{\gamma}$, then $\tilde{\delta} \in P(L_{m_0}^n)$, i.e., $p(m_0, n)$ is a critical cardinality. Therefore, if $\|\tilde{\alpha}\| = k + 1$, then the equality $\alpha_{n+1} = \alpha_n = 1$ is the necessary and sufficient condition for $p(m_0, n)$ to be a critical cardinality. Since $m_0 + 1$ must be a noncritical cardinality, then $\alpha_{n-2} = \alpha_{n-3} = 0$, and so the vector $\tilde{\alpha}$ must have the form

$$\tilde{\alpha} = (\alpha_1, ..., \alpha_{n-j-3}, 1, \underbrace{0, ..., 0}_{i}, 1, 1),$$

where $2 \leq j \leq n-3$. Then the numbers $m_0 + 1, m_0 + 2, ..., m_0 + j - 1$ are singular cardinalities. The number of such cardinalities equals $\sum_{i=2}^{n-2} (j-1) \cdot 2^{n-j-3} \simeq 2^{n-3}$.

If now $\|\alpha\| = k$, i.e., $m_0 = \sum_{i=0}^k \binom{n}{i}$, then the numbers $m_0 + j$, j = 1, ..., n - k, are candidates for singular cardinalities. The number of such candidates is no more than $\binom{n-1}{2} = o(2^{n-3})$.

 $\binom{n-1}{2} = o(2^{n-3}).$ The goal of the subsequent considerations is to obtain a necessary and sufficient condition for $\mathcal{K}_m^n \neq \emptyset$.

We say that a set B is *conjugate* to a set $A \subseteq B^n$ if $B = B^n \setminus P(A)$, and we write $B = A^*$.

Lemma 4.2 The set A^* is critical set for any $A \subseteq B^n$.

Proof.

Assume on the contrary that $\tilde{\alpha} \in S(A^*) \neq \emptyset$, i.e., $S_1^n(\tilde{\alpha}) \cap P(A^*) = \emptyset$. Note, that $\tilde{\alpha} \notin P(A)$. Then $\tilde{\alpha} \in \Gamma(A) \setminus S(A)$, since otherwise one has $S_1^n(\tilde{\alpha}) \cap P(A) = \emptyset$, i.e., $S_1^n(\tilde{\alpha}) \subseteq A^*$ and $\tilde{\alpha} \in P(A^*)$. Now, if for some vector $\tilde{\beta} \in S_1^n(\tilde{\alpha}) \cap A \neq \emptyset$ the inclusion $\tilde{\beta} \in \Gamma(A^*)$ holds, then $\tilde{\beta} \in \Gamma(A)$, so $\tilde{\alpha} \in P(A)$, i.e., $\tilde{\alpha} \notin A^*$. A contradiction.

Corollary 4.2.1 $\Gamma(A) \setminus S(A) = \Gamma(A^*)$.

Corollary 4.2.2 A set A is critical iff $(A^*)^* = A$.

A set $B \subseteq B^n$ is said to be *P*-optimal if it has the smallest possible number of boundary points among all sets with the same number of inner points. Denote by g(m, n) the number of boundary points of an *P*-optimal set $B \subseteq B^n$ with P(B) = m.

Theorem 4.2 For the existence of an m-element optimal critical set $A \subseteq B^n$ it is necessary and sufficient that $m - p(m, n) = g(2^n - m, n)$.

Proof.

If A is an optimal critical set, then $|\Gamma(A)| = m - p(m, n)$, $\Gamma(A) = \Gamma(A^*)$, $B^n \setminus A = P(A^*)$, $|P(A^*)| = 2^n - m$ and A^* is a P-optimal set, which implies the necessity of the condition in the Theorem.

If B is a P-optimal set and $|P(B)| = 2^n - m$, then $\Gamma(B) = \Gamma(B^*)$, $B^n \setminus P(B) = B^*$, $|B^*| = m$ and B^* is an optimal critical set.

Therefore, the problem of finding optimal critical sets reduces to the problem of finding P-optimal sets and conversely. In fact, the both problems are equally complex, as the results in the next Section show. However, in some cases, especially for small n, Theorems 4.1 and 4.2 are very useful for finding all optimal critical sets.

5 Construction of sets in the class \mathcal{K}_m^n

Let $A \subseteq B^n$ be an arbitrary critical subset, optimal or nonoptimal. We use it to construct a set $B \subseteq B^{n+1}$ as follows. Let us partition the cube B^{n+1} into two n-cubes $x_{n+1} = 0$ and $x_{n+1} = 1$. We construct the set A in the subcube $x_{n+1} = 0$ by setting the (n + 1)-st coordinate to 0 for each point $\tilde{\alpha} \in A$, and we construct the set P(A) similarly in the subcube $x_{n+1} = 1$. Let ψ denotes this transformation of the set A into B, i.e., $B = \psi(A)$ and let $\psi^s(A) = \psi(\psi^{s-1}(A)), \psi^0(A) = A$. Using the induction on s it is easy to show that if $A \subseteq B^n$ is a critical set, then $\psi^s(A)$ is a critical subset of the cube of dimension n + s. The main result of this Section is the following theorem.

Theorem 5.1 For any critical set $A \subseteq B^n$ there exists the number t(A) such that for any $t \ge t(A)$ the set $\psi^t(A) \subseteq B^{n+t}$ is a critical optimal set.

In order to proof the Theorem we need three auxiliary propositions.

Lemma 5.1 $\psi(L_m^n)$ is a standard arrangement.

Proof.

We use induction on n. For n = 2 the Lemma may be simply verified. Let us proceed the inductive step. Denote $A = \psi(L_m^n)$. Then $A^0(n+1)$ and $A^1(n+1)$ are standard arrangements in the respective subcubes $x_{n+1} = 0$ and $x_{n+1} = 1$. Moreover, $|A^0(n+1)| \ge |A^1(n+1)|$. Note that $|A^{00}(n+1,j)| \ge |A^{01}(n+1,j)|$ for the parts of $A^0(n+1)$ obtained when the subcube x_{n+1} is partitioned with the respect to an arbitrary coordinate x_j . We show that $A^0(j)$ and $A^1(j)$ are standard arrangements.

If x_j is a coordinate such that $p(|A^{00}(n+1,j)|, n-1) \le |A^{01}(n+1,j)|$ then by Lemma 3.2

$$P(A^{0}(n+1)) = P(A^{00}(n+1,j)) \cup P(A^{01}(n+1,j)).$$

Consequently,

$$P(A^{00}(n+1,j)) = \pi_{n+1}(A^{10}(n+1,j)),$$

and

$$P(A^{01}(n+1,j)) = \pi_{n+1}(A^{11}(n+1,j))$$

Then $A^0(j)$ and $A^1(j)$ are standard arrangements by the inductive hypothesis.

If x_j is a coordinate such that $p(|A^{00}(n+1,j)|, n-1) > |A^{01}(n+1,j)|$, then

$$P(A^{0}(n+1)) = \pi_{j}(A^{01}(n+1,j)) \cup P(A^{01}(n+1,j)),$$

which gives us that $A^{01}(n+1,j) = \pi_{n+1,j}(A^{10}(n+1,j))$, i.e., $A^{0}(j)$ is a standard arrangement. Since $\pi_{n+1}(A^{i}(n+1)) = P(A^{0}(n+1))$, then $P(A^{01}(n+1,j)) = \pi_{n+1}(A^{11}(n+1,j))$ and hence $A^{1}(j)$ is a standard arrangement.

Therefore, $A^0(j)$ and $A^1(j)$ are standard arrangements for any $j, 1 \le j \le n+1$, from which by Lemma 3.6a) and the equality $\pi_{n+1}(A^1(n+1)) = P(A^0(n+1))$ we get A is a standard arrangement.

Lemma 5.2 Let $A \subseteq B^n$ be an optimal set. Then $B = \psi^t(A)$ is also an optimal set for any $t \ge 1$.

Proof.

It is sufficient to consider the case t = 1. Denote |A| = m. Then

$$|B| = m + |P(A)|, \quad |P(B)| = |P(A)| + |P(P(A))|.$$

By Theorem 3.1,

$$|B| = m + p(m, n), \quad |P(B)| = p(m, n) + p(p(m, n), n).$$

So in order to complete the proof we have to show that

$$p(m + p(m, n), n + 1) = p(m, n) + p(p(m, n), n).$$

For this we consider the set L_m^n . Denote $C = \psi(L_m^n)$. By Lemma 5.1, $C = L_{m+p(m,n)}^{n+1}$. By the definition of ψ , $C^0(n+1)$ and $C^1(n+1)$ are standard arrangements, i.e., optimal subsets in n dimensions. Since $P(C^0(n+1)) = \pi_{n+1}(C^1(n+1))$ by the definition of ψ , then

$$|P(C)| = p(m + p(m, n), n + 1) = |C^{1}(n + 1)| + |P(C^{1}(n + 1))| = p(m, n) + p(p(m, n), n),$$

and the Lemma follows.

Lemma 5.3 Let A be a critical set, such that P(A) is an optimal set. Then there exists a number t(A), such that $B = \psi^{t(A)}(A)$ is an optimal set.

Proof.

Note, that not for any number q there exists an optimal set with exactly q inner points. Indeed, for example,

$$p\binom{n}{2} + n, n = n - 1$$
 and $p\binom{n}{2} + n + 1, n = n + 1.$

Since p(m, n) is nondecreasing on m for fixed n, there does not exist an optimal subset of B^n with exactly n inner points. Let |A| = m. Corresponding to the remark just made, we consider two cases.

Case 1. Suppose that there exists an optimal subset of B^n with exactly |P(A)| inner points. Denote by \overline{m} the greatest number a such that p(a, n) = |P(A)|, and by $\psi^s(m)$ the cardinality of the set $\psi^s(A)$, s = 1, 2, ... Note that A is an optimal set iff $\overline{m} \ge m$. Taking into consideration Lemma 5.2, we may assume that A is nonoptimal set, i.e., $m - \overline{m} > 0$.

a) Assume \overline{m} is noncritical cardinality. Consequently, $\overline{m} + 1$ is a critical cardinality, from where, taking into consideration

$$|\psi(L^n_{\overline{m}})| = \overline{m} + |P(A)|, \quad \psi(L^n_{\overline{m}}) = L^{n+1}_q$$

and

$$|L_q^{n+1} \cap \{(x_1, ..., x_n, 0) \in B^{n+1}\}| = m,$$

where $q = \overline{m} + |P(A)|$, one gets

$$|P(L_{q+1}^{n+1})| = |P(L_q^{n+1})|$$
 and $|P(L_{q+2}^{n+1})| > |P(L_{q+1}^{n+1})|.$

Therefore, $\psi(m) = \overline{m} + |P(A)| + 1$. Since $\psi(m) = m + |P(A)|$, then $P(\psi(A))$ is an optimal set by Lemma 5.2. From $|P(L_q^{n+1})| = |P(\psi(A))|$ it follows that there exists an optimal subset $C' \subseteq B^{n+1}$ with $|P(C')| = |P(\psi(A))|$. Finally, the equalities $\overline{\psi(m)} = q + 1$ and $|P(L_{q+1}^{n+1})| = |P(L_q^{n+1})|$ imply that $\overline{\psi(m)}$ is a noncritical cardinality (in n+1 dimensions). Consequently, applying the arguments from above to the sets $\psi(A)$, $\psi^2(A)$, ..., we get that

$$m - \overline{m} - 1 = \psi(m) - \overline{\psi(m)}, \quad \psi^{s-1}(m) - \overline{\psi^{s-1}(m)} - 1 = \psi^s(m) - \overline{\psi^s(m)},$$

i.e., $m - \overline{m} - s = \psi^s(m) - \overline{\psi^s(m)}$. For s sufficiently large, the right hand side of the last equality is negative, i.e., $\psi^s(A) \subseteq B^{n+s}$ is an optimal set.

b) Assume now that \overline{m} is a critical cardinality. Since $p(\overline{m}+1,n) > p(\overline{m},n)$ then $\overline{m}+1$ is also a critical cardinality. Denote by $\tilde{\alpha}(L_m^n)$ the greatest vector of L_m^n in order \mathcal{L} . From Lemma 3.7 it follows that

$$\tilde{\alpha}(L_m^n) = (\alpha_1, \dots, \alpha_{n-r-2}, 1, 0, \underbrace{1, \dots, 1}_r),$$

where $r \ge 1$. Now we show that $\overline{\psi(m)} = \overline{m} + |P(A)| + 1$. For this consider the set $\psi(L_{\overline{m}}^n)$. We have that $\psi(L_{\overline{m}}^n) = L_q^{n+1}$ by Lemma 5.1 and

$$\tilde{\alpha}(L_q^{n+1}) = (\alpha_1, ..., \alpha_{n-r-2}, 1, 0, 0, \underbrace{1, ..., 1}_r),$$

for $q = \overline{m} + |P(A)|$. Since

$$\tilde{\alpha}(L_{q+1}^{n+1}) = (\alpha_1, ..., \alpha_{n-r-2}, 0, \underbrace{1, ..., 1}_{r+1})$$

and

$$\tilde{\alpha}(L^{n+1}) = (\alpha_1, ..., \alpha_{n-r-2}, 0, \underbrace{1, ..., 1}_r, 0, 1),$$

then q+1 is noncritical cardinality in B^{n+1} and q+2 is a critical cardinality. Consequently,

$$p(q, n+1) = p(q+1, n+1) < p(q+2, n+1)$$

and hence $\overline{\psi(m)} = \overline{m} + |P(A)| + 1$. Now from $P(\psi(A)) = \psi(P(A))$ and Lemma 5.2 it follows that $P(\psi(A))$ is an optimal set. Since $|P(\psi(A))| = |P(L_{q+1}^{n+1})|$ and $\overline{\psi(m)}$ is noncritical cardinality, the proof of the lemma can be reduced to Case 1a) by applying the arguments there to the set $\psi(A)$.

Case 2. Assume now that there does not exist an optimal subset of B^n with exactly |P(A)| inner points. Now we show that in B^{n+1} there exists an optimal set with exactly $|P(\psi(A))|$ inner points. This will reduce the proof of the lemma in Case 2 to Case 1.

Denote |P(A)| = t and let us show that t + 1 is a critical cardinality in B^n . Indeed, p(s,n) > t > p(s-1,n) for some number s. Let q = p(s,n) and $v = q - t \ge 1$. Then s is a critical cardinality in B^n and $p(s,n) - p(s-1,n) \ge 2$. Consequently

$$\tilde{\alpha}(L_s^n) = (\alpha_1, \dots, \alpha_{n-r-2}, 1, 0, \underbrace{1, \dots, 1}_r),$$

and so

$$\tilde{\alpha}(L_t^n) = (\alpha_1, ..., \alpha_{n-r-2}, 1, 0, \underbrace{\underbrace{1, ..., 1}_v, 0, 1, ..., 1}_v).$$

Therefore, t + 1 is a critical cardinality and for the reduction it is sufficient to show that $|P(\psi(A))| + 1$ is noncritical cardinality in B^{n+1} .

Since $P(\psi(A)) = \psi(P(A))$ and $\psi(P(A))$ is an optimal set by Lemma 5.2, then $\psi(L_t^n) = L_w^{n+1}$ for $w = |\psi(P(A))|$. But then

$$\tilde{\alpha}(L_w^{n+1}) = (\alpha_1, ..., \alpha_{n-r-2}, 1, 0, \underbrace{\underbrace{1, ..., 1}_{v}, 0, 0, 1, ..., 1}_{v}),$$

i.e. w + 1 is noncritical cardinality, which competes the proof.

Proof of Theorem 5.1. Denote by d(A) the greatest number d, such that $P_d(A) \neq \emptyset$. If $P(A) = \emptyset$, then let d(A) = 0.

Using induction on s, it is easy to show $P_s(\psi(A)) = \psi(P_s(A))$, from which $d(\psi(A)) = d(A)$ follows. Note that \emptyset is an optimal set and $P_{d(A)+1}(A) = \emptyset$. Denote by $r \ge 1$ the smallest integer for which $P_r(A)$ is an optimal set. Then $r \le d(A) + 1$. Let $C = P_{r-1}(A)$. Applying to C sufficiently many (n_1) times transformation ψ , we get that $\psi^{n_1}(C) = P_{r-1}(\psi^{n_1}(A))$ is an optimal set. Analogously, applying to A sufficiently many (n_2) times

transformation ψ , we get that $P_{r-2}(\psi^{n_1+n_2}(A))$ is an optimal set and so on r times. Since $d(\psi^t(A)) = d(A)$ and $r \leq d(A) + 1$, then $t(A) = n_1 + n_2 + \cdots + n_r$ is a finite number and $\psi^{t(A)}(A)$ is an optimal critical set (in n + t(A) dimensions). Finally, for $t \geq t(A)$ the Theorem follows from Lemma 5.2.

Note that $\psi^{t(A)}(A)$ is an optimal set of critical cardinality iff A is an optimal set of critical cardinality. Indeed, if |A| = m is a critical cardinality and m + |P(A)| = t, then $\psi(L_m^n) = L_t^{n+1}$ by Lemma 5.1. But L_m^n is a critical set, so L_t^{n+1} is a critical set too, i.e. t is a critical cardinality.

On the other hand, let $\psi^{t(A)}(A) = B$ be an optimal set. If |B| is a critical cardinality, then $p(|B^0(n+1)|, n) \ge |B^1(n+1)|$, since $B^1(n+1) = \pi_{n+1}(P(B^0(n+1)))$. If $p(|B^0(n+1)|, n) = |B^1(n+1)|$, then $B^0(n+1)$ is an optimal set by Lemma 3.2b). We replace the sets $B^0(n+1)$ and $B^1(n+1)$ by standard arrangements (in *n* dimensions). It follows from Lemma 5.1, that the resulting set is a standard arrangement in B^{n+1} . Since |B|is a critical cardinality, then $|B^1(n+1)|$ is also a critical cardinality. Therefore, since $B^0(n+1)$ is a critical set, then $|B^0(n+1)|$ is a critical cardinality by Lemma 4.1.

But if $p(|B^0(n+1)|, n) > |B^1(n+1)|$, then $B^0(n+1)$ is an optimal set of critical cardinality (see [1], Remark 6.2 on page 117).

Therefore, if A is an optimal critical set and $A \notin \mathcal{K}_m^n$ (i.e. |A| is a critical cardinality), then $|\psi^t(A)|$ is a critical cardinality too for all $t, t \geq 1$, i.e., $\psi^t(A) \notin \mathcal{K}_{m'}^{n+t}$ (here $m' = |\psi(A)|$). But if A is nonoptimal set, or $A \in \mathcal{K}_m^n$, then $\psi^{t(A)}(A) \in \mathcal{K}_{m'}^{n+t(A)}$.

Corollary 5.1.1 Let $k(n) = |\{m : 1 < m < 2^n \& \mathcal{K}_m^n \neq \emptyset\}|$. Then $k(n+1) \ge k(n)$.

It follows from Theorem 5.1, that the collection of optimal critical sets of noncritical cardinality distinguishes itself by its great diversity. In order to demonstrate this more clearly we consider the following construction.

Let $A \subseteq B^n$ be an arbitrary critical set, not necessary optimal. Note that such a set may be represented (not uniquely) as a union of *n*-dimensional balls with radii at least one, i.e.

$$A = S_{r_1}^n(\tilde{\alpha}_1) \cup \dots \cup S_{r_t}^n(\tilde{\alpha}_t), \quad 1 \le r_i \le n, \ 1 \le i \le t.$$

The simplest example of such representation is $A = \bigcup_{\tilde{\alpha} \in P(A} S_1^n(\tilde{\alpha})$. Moreover, any set that is a union of such balls is critical.

We fix one of such representations of a critical set $A \subseteq B^n$ and construct a set $B \subseteq B^{n+1}$ as follows. Let the cube B^{n+1} be partitioned into two *n*-dimensional subcubes $x_{n+1} = 0$ and $x_{n+1} = 1$. In the subcube $x_{n+1} = 1$ we construct points $\tilde{\beta}_i$ $(1 \le i \le t)$ by setting the (n+1)-st coordinate of $\tilde{\beta}_i$ equal to 0, and the remaining *n* coordinates equal to the same as for point $\tilde{\alpha}_i$. Let

$$B = S_{r_1}^{n+1}(\tilde{\beta}_1) \cup \dots \cup S_{r_t}^{n+1}(\tilde{\beta}_t) \subseteq B^{n+1}.$$

The structures of sets A and B have much in common. In fact B is obtained by "extending" A in a space of large dimensions. It is useful to compare the similarity between A and B with the similarity between the balls $S_r^n(\tilde{\alpha}) \subseteq B^n$ and $S_r^{n+1}(\tilde{\beta}) \subseteq B^{n+1}$. Denote by τ the transformation of A into B, and let $\tau^s(A) = \tau(\tau^{s-1}(A))$ and $\tau^0(A) = A$. Note, that if $A \subseteq B^n$ is a critical set, then for any $t \ge 1$, the set $\tau^t(A) \subseteq B^{n+t}$ is also a critical set. **Corollary 5.1.2** For any critical set $A \subseteq B^n$ there exists a number q(A), such that for any $t \ge q(A)$ the set $\tau^t(A)$ is optimal and critical (in n + t dimensions).

Proof.

It is sufficient to show that if $A \subseteq B^n$ is a critical set, $B = \tau^t(A)$ and $t \ge d(A)$, then $\psi(B) = \tau(B)$, since then we may apply Theorem 5.1.

By the definition of τ , there exist balls $S_{r_i}^n(\tilde{\alpha}_i)$ such that

$$A = \bigcup_{i=1}^{s} S_{r_i}^n(\tilde{\alpha}_i), \quad \text{and}B = \bigcup_{i=1}^{s} S_{r_i}^{n+t}(\tau(\tilde{\alpha}_i))$$

It is not hard to show that $\psi(A) = \tau(A)$ iff

$$P(A) = \bigcup_{i=1}^{s} P(S_{r_i}^{n+t}(\tau(\tilde{\alpha}_i)))$$
(1)

If this inequality holds for A, then we pass to the consideration of the set $\tau(A)$. If (1) does not hold, then there exists a point $\tilde{\gamma}$ of $\psi(A)$ such that

$$\tilde{\gamma} = (\gamma_1, ..., \gamma_n) \in P(A) \setminus \bigcup_{i=1}^s P(S_{r_i}^n(\tilde{\alpha}_i).$$

Then the (n+1)-th entry of $\tau(\tilde{\gamma})$ equals 0 and $\pi_{n+1}(\tilde{\gamma}) \notin \tau(A)$.

Further, if (1) holds for $\tau(A)$, then we pass to the consideration of the set $\tau^2(A)$. But if (1) does not hold, then there exists a point $\tilde{\gamma}_1 \in \tau(A)$ such that

$$\tilde{\gamma}_1 \in P(\tau(A)) \setminus \bigcup_{i=1}^{\circ} P(S_{r_i}^{n+1}(\tau(\tilde{\alpha}))).$$

Then the (n + 1)-th entry of $\tilde{\gamma}_1$ equals 1 and $\pi_{n+1}(\tilde{\gamma}_1) \in P(A)$.

Analogously, if (1) does not hold for the set $\tau^{q}(A)$, then the coordinates of

$$\tilde{\gamma}_q \in P(\tau^q(A)) \setminus \bigcup_{i=1}^s P(S^{n+q}_{r_i}(\tilde{\beta}))$$

with indices n + 1, ..., n + q are equal to 1 and $\pi_{n+1,...,n+q}(\tilde{\gamma}_q) \in P_q(A)$. To complete the proof it remains to observe that $P_q(A) = \emptyset$ for $q \ge d(A)$.

Theorem 5.1 and Corollary 5.2 can be regarded as ways of constructing optimal sets in the class \mathcal{K}_m^n .

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