Minimization of Surrounding of Subsets in Hamming Space

SERGEI L. BEZRUKOV

1 Introduction

As it is well-known, the classical isoperimetric problem on the plane claims to find a simple closured curve of the plane that bounds a maximal possible area. We shell concern with some discrete analog of this problem. It was proved in [1], that for any integer m, represented in the form $m = \sum_{i=0}^{r} \binom{n}{i} + \delta$, $0 \le \delta < \binom{n}{r+1}$, one of m-element subsets on n-cube B^n with minimum boundary is the union of the ball of radius r centered in the origin $(0, ..., 0)$ united with the final lexicographical segment of length δ (see below).

In Section 2 we are concerned with the isoperimetric type problem for the even levels of B^n (i.e. the collection of all vertices α of B^n with $\|\alpha\| \equiv 0 \pmod 2$) and show that a ball in the even levels of B^n united with a final lexicographical segment (i.e. a similar construction) is a solution of it.

Section 3 of this paper is devoted to isoperimetric type problems for B^n with some restrictions on the boundary operator.

Denote by $S_k^n(\alpha)$ the Hamming ball of radius k centered in the point $\alpha \in B^n$. Let $A \subseteq B^n$. We call a point $\alpha \in A$ the boundary point of A iff $S_1^n(\alpha) \notin A$. The collection of all boundary points of A is called the boundary of A and denoted by $\Gamma(A)$. The subset

$$O(A) = \{\alpha \in B^n \backslash A : S_1^n(\alpha) \cap A \neq \emptyset\}$$

is called the surrounding of a set $A \subseteq B^n$. Obviously

$$O(A) = \Gamma(B^n \backslash A).$$

Denote by B_k^n the k-th level of B^n, i.e. the collection of all vertices of B^n exactly with k nonzero coordinates. For a point $\alpha = (\alpha_1, ..., \alpha_n) \in B^n$ the number

$$l(\alpha) = \sum_{i=1}^{n} \alpha_i \cdot 2^{n-i}$$

is called the lexicographical number of α. For a given k we call the subset $L_k^n(m) \subseteq B_k^n$ the final lexicographical segment if it consists of m points of B_k^n with last m maximal lexicographical numbers.

Finally for $\alpha \in B^n$ denote by $\pi_i(\alpha)$ the operator of projecting, which inverts the i-th entry of α and for $A \subseteq B^n$ let $\pi_i(A) = \bigcup_{\alpha \in A} \pi_i(\alpha)$.

2 Isoperimetric problem for the even levels of B^n

Denote

$$B^{n,0} = \{(\alpha_1, ..., \alpha_n) \in B^n : \alpha_1 + ... + \alpha_n \equiv 0 \pmod 2\}$$

and consider the problem to find an m-element subset $A^* \subseteq B^{n,0}$ with minimal cardinality of the surrounding among all the m-element subsets of $B^{n,0}$, for $m = 1, ..., 2^{n-1}$.

It is easy to see that an integer $m \in [0, 2^{n-1}]$ may be uniquely represented in the form

$$m = \sum_{i=0}^{l} \binom{n}{2i} + \delta, \quad 0 \le \delta < \binom{n}{2l+2}.$$

Denote

$$M^n(m) = B_0^n \cup B_1^n \cup \cdots \cup B_{2l}^n \cup L_{2l+2}^n(\delta).$$

Theorem 1 *$|O(A)| \ge |O(M^n(m))|$ for any $A \subseteq B^{n,0}$, $|A| = m$.*

Proof.
Split B^n into two parts B_0 and B_1 by the first entry. We shell consider these parts as $(n-1)$-dimensional subcubes. Denote

$$A_0 = A \cap B_0 \quad \text{and} \quad A_1 = A \cap B_1.$$

Divide $O(A)$ into two parts $Q(A)$ and $R(A)$, where

$$\begin{aligned} R(A) &= \{\alpha \in O(A) \cap B_0 : S_1^{n-1}(\alpha) \cap A_0 \neq \emptyset \text{ and } \pi_1(\alpha) \notin A_1\} \cup \\ &\quad \{\alpha \in O(A) \cap B_1 : S_1^{n-1}(\alpha) \cap A_1 \neq \emptyset \text{ and } \pi_1(\alpha) \notin A_0\}, \\ Q(A) &= O(A) \setminus R(A). \end{aligned}$$

Notice that $|Q(A)| = |A|$. Indeed, for $\alpha \in A$ consider $\beta = \pi_1(\alpha)$. Then $\beta \in Q(A)$, which implies $|Q(A)| \ge |A|$. The inverse inequality may be proved similarly.

Therefore, $|O(A)| = |R(A)| + |A|$ and in order to minimize $|O(A)|$ it is sufficient to minimize $|R(A)|$. For this purpose denote by C the set obtained from A by replacing of each point $\alpha \in A_1$ to $\pi_i(\alpha)$. Then $|C| = |A|$ and it is easy to show, that $|R(A)| = |R(C)|$.

Indeed, if for example $\alpha \in B_0 \cap R(A)$, then $\alpha \in R(C)$ either, and if $\alpha \in B_1 \cap R(A)$, then $\pi_i(\alpha) \in R(C)$. Hence, $|R(A)| \le |R(C)|$. The other inequalities may be proved similarly.

Since $|Q(A)| = |Q(C)|$, then $|O(A)| = |O(C)|$. On the other hand, by a given set $C \subseteq B_0$ we may construct the set $A \subseteq B^{n,0}$, such that $|Q(A)| = |Q(C)|$ holds. Therefore, in order to solve our problem it is necessary and sufficient to construct an m-element subset $C \subseteq B_0$ with minimum value of $|O(C)|$.

Represent now the number m in the form

$$m = \sum_{i=0}^{k} \binom{n-1}{i} + \epsilon, \quad 0 \le \epsilon < \binom{n-1}{k+1}$$

and denote

$$H^{n-1}(m) = S_k^{n-1}(\mathbf{0}) \cup L_{k+1}^{n-1}(\epsilon).$$

By [1], the subset $H^{n-1}(m)$ has minimal surrounding among all the m-element subsets of B^{n-1}, which completes the proof of the Theorem. ∎

Remark 1 *The problem to construct an m-element subset of $B^n \setminus B^{n,0}$ with minimal surrounding may be solved similarly.*

3 Isoperimetric type problems for different boundary operators

We say that an edge $(\alpha, \beta) \in E(B^n)$ is of i-th direction if α and β differ in the i-th entry. Let $M \subseteq \{1, ..., n\}$ be a fixed subset and $|M| = t$. For $A \subseteq B^n$ and $\alpha \in A$ denote

$$\begin{aligned} O(\alpha, M) &= \{\beta \in O(\alpha) : (\alpha, \beta) \text{ is of } i\text{-th direction for some } i \in M\}, \\ O(A) &= \bigcup_{u \in A} O(\alpha, M). \end{aligned}$$

For given m and M consider the problem of finding an m-element subset $A \subseteq B^n$ with minimal value of $|O(A, M)|$ among all the m-element subsets of B^n.

Represent the number m in the form $m = p \cdot 2^t + q$, $0 \leq q < 2^t$. Consider the subspace B_0 of the n-cube with basis M and denote by B_i, $1 \leq i \leq 2^{n-t} - 1$, the shifts of B_0 on vectors from the orthogonal subspace $B_0^{\perp}$. Denote

$$A^* = B_0 \cup B_1 \cup \cdots \cup B_{p-1} \cup H^t(q),$$

where $H^t(q) \subseteq B_p$.

Theorem 2 $|O(A^*, M)| \leq |O(A, M)|$ *for any* $A \subseteq B^n$, $|A| = m$.

Proof.
Let $A \subseteq B^n$. Denote $A_i = A \cap B_i$, $1 \leq i \leq 2^{n-t} - 1$. Then $O(A, M) = \bigcup_{i=0}^{2^{n-t}} O(A_i)$, where the operator O in the right hand side is applied in B_i only. Replace each A_i to the set $H^t(|A_i|)$ in B_i. We obtain a set $C \subseteq B^n$, such that $|O(C, M)| \leq |O(A, M)|$. If $C_i \neq \emptyset$ for the only i, then the Theorem is true. Let $C_i \neq \emptyset$, $C_j \neq \emptyset$ and $B_i = B_0 \oplus \alpha$, $B_j = B_0 \oplus \beta$, where $\oplus$ denotes the modulo 2 addition (or the shift). Consider the two cases:

Case 1. Assume $|C_i| + |C_j| = r \leq 2^t$. Denote $\bar{C} = C \oplus \mathbf{1}$, $\mathbf{1} = (1, ..., 1) \in B_j$ and let $C_j' = C_j \oplus \alpha \oplus \beta \subseteq B_i$. Then $C \cap_i \cap C' = \emptyset$, but $|O(H^t(r))| \leq |O(C_i \cup C')| \leq |O(C_i)| + |O(\bar{C})| = |O(C_i)| + |O(C_j)|$. Replace C_i by $H^t(r)$ and replace C_j by $\emptyset$.

Case 2. Let $r > 2^t$. Assume without loss of generality that $|C_i| < 2^t$ and $|C_j| < 2^t$. Denote $\bar{C}_i = B_i \setminus C_i$, $\bar{C}_j = B_j \setminus C_j$. Then $|\bar{C}_i| + |\bar{C}_j| = s \leq 2^t$, and $|O(C_i)| = |\Gamma(\bar{C}_i)|$, $|O(C_j)| = |\Gamma(\bar{C}_j)|$ (here operators O and Γ are applied in t dimensions). Now consider $C' = \bar{C}_j \oplus \mathbf{1} \in B_j$ and let $C'' = C' \oplus \alpha \oplus \beta \subseteq B_i$. Then $C' \cap C'' = \emptyset$, but $|O(H^t(2^t - s))| = |\Gamma(H^t(s))| \leq |\Gamma(C' \cup C'')| = |\Gamma(\bar{C}_i)| + |\Gamma(C)| = |\Gamma(\bar{C}_i)| + |\Gamma(\bar{C}_j)| = |O(C_i)| + |O(C_j)|$. Replace now C_i by B_i and replace C_j by $H^t(r - 2^t)$.

Therefore in the both cases we obtained a set D, such that $|O(D, M)| \leq |O(A, M)|$. It is easy to see that using the similar transformations we may get the set A^* from A without increasing the value of $|O(A, M)|$. ∎

Remark 2 *The following slight generalization of this problem is possible. Let M be a collection of linearly independent over modulo 2 addition vectors of B^n. If we denote $O(\alpha, M) = M \oplus \alpha$ then the problem of constructing of an m-element subset $A \subseteq B^n$ with minimal value of $O(A, M)$ may be solved similarly.*

References

[1] Harper L.H. *Optimal numberings and isoperimetric problems on graphs*, J. Comb. Theory **1**(1966), No 1, p.385–393.