Specification of all Solutions of the Discrete Isoperimetric Problem that Have a Critical Cardinality

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Extended abstract

We present here a description of all solutions of the isoperimetric problem in Hamming space of some special cardinalities. The number of these cardinalities equals 2^{n-1}.

Let B^n denotes the vertex set of the $n-$dimensional unit cube with Hamming metric and $A \subseteq B^n$. Denote by $S_k^n(\alpha)$ the sphere of radius k centered in $\alpha \in B^n$. We call a point $\alpha \in A$ the inner point of a set A if $S_1^n(\alpha) \subseteq A$ and the boundary point of A in the opposite case. Denote by $P(A)\Gamma(A)$ the collection of all inner (boundary) points of A.

Consider an isoperimetric problem to find for a fixed m, $1 \leq m \leq 2^n$, an m-element set $A \subseteq B^n$, such that $|\Gamma(A)| \leq |\Gamma(B)|$ for any $B \subseteq B^n$, $|B| = m$. We call such a set A the optimal one. In [2] it is shown that the set $L(n,m)$ is a solution of the isoperimetric problem. $L(n,m)$ is defined as the initial segment of length m of the following order of vertices of B^n. We say that a vertex $\alpha \in B^n$ precedes $\beta \in B^n$ iff $\|\alpha\| \leq \|\beta\|$, or if $\|\alpha\| = \|\beta\|$ then α is greater β in the lexicographical order, where $\|\alpha\|$ is the coordinate sum of α.

For $A \subseteq B^n$ we call a point $\alpha \subseteq A$ the free point of A if $P(A) = P(A\backslash\alpha)$ and denote by $S(A)$ the collection of them. If $S(A) = \emptyset$, then a set A is called critical.

Lemma 1 *Let A is an optimal noncritical set and α is it's free point. Then the set $A\backslash\alpha$ is optimal either and $S(A\backslash\alpha) = S(A)\backslash\alpha$.*

Corollary 1 $|S(A)| \leq |S(L(n,|A|))|$.

Corollary 2 *$L(n,m)$ is a critical set then any optimal m-element subset of B^n is critical too.*

We call a number m critical cardinality if $L(n,m)$ is a critical set.

Let m and n be fixed and m^* be the greatest critical cardinality less or equal to m. It follows from Lemma 1 that if A is an optimal critical r-element subset, $m^* \leq r \leq m$, then adding to it arbitrary $m-r$ points from $B^n\backslash A$ we get an optimal m-element set. Therefore the description of all optimal subset of B^n is reduced to the description of all critical optimal sets only.

A specification of all optimal subsets meets a lot of difficulties. For example it turned out that in some cases the set $P(A)$ may be unconnected. Indeed, if there exists an integer solution of equation $\Gamma(n,x)+\Gamma(n,m-x)=\Gamma(n,m)$ ($\Gamma(n,m)=|\Gamma(A)|$ for an optimal m-element set $A\subseteq B^n$)), then the set $A=B\cup C$ for $B=L(n,m-x)$, $C=B^n\backslash L(n,2^n-x)$ is optimal. Moreover, since $B\cap C=\emptyset$ then at least for $m\leq 2^{n-1}$ there is a wide freedom for embedding of these parts into B^n. However, if we choose optimal sets with another structure as B and C then it is not clear whether it is possible to embed them together in B^n with empty intersection. Similar situation occurs when A may be divided to $t\geq 2$ nonintersecting pieces. A necessary condition of such division is the existence of integer solutions of equation $\Gamma(n,x_1)+\Gamma(n,x_2)+\cdots+\Gamma(n,x_t)=\Gamma(n,m)$ under condition $x_1+\cdots+x_t=m$.

There are examples of m-element optimal sets which are either unconnected or connected for the same m, and at the last case their structure is not similar to the structure of $L(n,m)$ in the sense that they cannot be obtained from $L(n,m)$ by means of isometric transformations of B^n.

This paper is devoted to the specification of the family of optimal subsets of critical cardinalities.

Let us split the n-cube by the i-th coordinate x_i into two $(n-1)$-cubes and denote by $A^0(i)$ and $A^1(i)$ the parts of a set $A\subseteq B^n$ in these subcubes. We say that A is i-normalized if $|A^0(i)|\geq|A^1(i)|$. A set which is i-normalized for $i=1,...,n$ is called simply normalized. Denote by $K(n,m)$ the collection of all m-element optimal subsets of critical cardinality m and by $\mathcal{K}(n,m)$ the collection of all normalized subsets in $K(n,m)$. It is clear that each subset $A\in K(n,m)\backslash\mathcal{K}(n,m)$ may be transformed to some $B\in\mathcal{K}(n,m)$ by "shifting" it coordinatewise modulo 2 on some binary vector γ. The i-th coordinate of γ equals 0 iff A is i-normalized and 1 otherwise.

Lemma 2 $|K(n,m)|\leq 2^n\cdot|\mathcal{K}(n,m)|$.

Let us assume that the number m is represented in the form $m=\sum_{i=0}^{k}\binom{n}{i}+\delta$ with $0\leq\delta<\binom{n}{k+1}$. We call a coordinate x_i minimal coordinate of a set $A\subseteq B^n$ if $\||A^0(i)|-|A^1(i)|\|\leq\||A^0(j)|-|A^1(j)|\|$ for any $j=1,...,n$. The following lemma is proved in [1].

Lemma 3 *Let $A\in\mathcal{K}(n,m)$ and x_i be it's minimal coordinate. Then $A^0(i)$ and $A^1(i)$ are optimal subsets (in $n-1$ dimensions) of critical cardinalities and*

$$|A^0(i)|=\sum_{i=0}^{k}\binom{n-1}{i}+\left(\delta\ominus\binom{n-1}{k}\right),\quad |A^1(i)|=\sum_{i=0}^{k}\binom{n-1}{i}-\left(\binom{n-1}{k}\ominus\delta\right),$$

where $a\ominus b$ equals $a-b$ if $a\geq b$ and 0 otherwise.

Our approach for the specification of all subsets of $\mathcal{K}(n,m)$ is in the following. We introduce the concept of a division process of a set $A\in\mathcal{K}(n,m)$. It consists of some steps and defines by induction on the number of step. On the first step we divide the set A by it's arbitrary minimal coordinate x_{i_1}. It follows from Lemma 3 that the set $A^0(i_1)$ (when $\delta<\binom{n-1}{k}$), or the set $A^1(i_1)$ (when $\delta\geq\binom{n-1}{k}$) is a sphere. The structure of the other set ($A^1(i_1)$ (resp. $A^0(i_1)$) is unclear yet. We call this set unknown. Assume that after $l-1$

($l \geq 2$) steps we have the only unknown set in some $(n-l+1)$-subcube and all the other parts of A in such subcubes are spheres. Then on the l-th step of our process we split B^n into $(n-l)$-subcubes by arbitrary minimal coordinate x_{i_l} of the unknown set. From Lemma 3 it follows again, that we obtain no more than one unknown set in such a way and the other parts of A in the corresponding $(n-l)$-subcubes are spheres. We call them l-spheres. The process terminates on the t-th step if the unknown set is a sphere.

Lemma 4 *Let $A \in \mathcal{K}(n,m)$. Then*

(i) There exist numbers $l_1, m_1, l_2, m_2, ..., l_r, m_r$ ($0 < l_1 \leq m_1 < l_2 \leq m_2, ..., < l_r \leq m_r$), such that δ may be uniquely represented in the following canonical form

$$\delta = \binom{n-l_1}{k-l_1+1} + \binom{n-l_1-1}{k-l_1+1} + \cdots + \binom{n-m_1}{k-l_1+1} +$$

$$\binom{n-l_2}{k-l_1-l_2+m_1+2} + \cdots + \binom{n-m_1}{k-l_1-l_2+m_1+2} +$$

$$\binom{n-l_r}{k-\sum_{i=1}^{r} l_i + \sum_{i=1}^{r-1} m_i + r} + \cdots + \binom{n-m_r}{k-\sum_{i=1}^{r} l_i + \sum_{i=1}^{r-1} m_i + r};$$

(ii) The number of steps in the splitting process equals m_r.

We call a sequence I of coordinates $x_{i_1}, ..., x_{i_m}$ admissible (here $m = m_r$) for a set $A \in \mathcal{K}(n,m)$ if there exists a splitting process, such that on the j-th step of it we split B^n by the coordinate x_j, for $j = 1, ..., m_r$. Denote the collection of admissible sequences for A by $I(A)$ and for $I \in I(A)$ let $U_t(A, I)$ be the unknown set obtained after t steps of the splitting process. Consider the class $\mathcal{L}(n,m)$ of subset $A \in \mathcal{K}(n,m)$, such that for all $t = 1, ..., m_r$ the set $U_t(A, I)$ is normalized (in $(n-t)$ dimensions) for some $I \in I(A)$. Denote by $\varphi(A)$ the set obtained from A by a permutation φ of coordinates of B^n.

Lemma 5

(i) If $A \in \mathcal{L}(n,m)$, then $A = \varphi(L(n,m))$ for some permutation φ;

(ii) $|\mathcal{L}(n,m)| = \frac{n!}{(l_1-1)!\cdot(m_1-l_1+1)!\cdots(l_r-m_r-1)!\cdot(m_r-l_r+1)!\cdot(n-m_r)!}$.

It turned out that that all the m_r-spheres (maybe except of $U_{m_r}(A, I)$) of $A \in \mathcal{K}(n,m)$ are centered at the origins of the corresponding $(n-m_r)$-subcubes. The center α of the sphere $U_{m_r}(A, I)$) is either at the origin of the corresponding subcube (point β) or at some point γ of norm 1 (in this subcube). Notice, that if $\alpha = \beta$ then $A \in \mathcal{L}(n,m)$. Denote by $L(A) \in \mathcal{L}(n,m)$ a set for which there exists a splitting process with the same sequence, i.e. $I(A) \cap I(L(A)) \neq \emptyset$. Then A may be obtained from $L(A)$ by some transposition of the m_r-spheres and maybe by replacing the center α of the sphere $U_{m_r}(A, I)$) to a point γ. Therefore, the specification of all the subsets from $\mathcal{K}(n,m) \backslash \mathcal{L}(n,m)$ may be reduced to specification of such transformations.

Let $\gamma \in B^n$. Denote by $C(\gamma, 1)$ the $(n - \|\gamma\|)$-subcube of B^n including the points γ and $\mathbf{1} = (1, ..., 1)$. We call two m_r-spheres S_1 and S_2 i-neighboring if the origins of the corresponding $(n-m_r)$-subcubes differ in the i-th entry only. Therefore for fixed i the set of m_r-spheres is divided into pairs of i-neighboring spheres. Define the transformation

$R(\alpha, x_i)A$ of a set $A \in \mathcal{K}(n,m)$, which is in the following (here $I = \{x_{i_1}, ..., x_{i_m} \in I(A)$ and $m = m_r$):

1). If $\|\alpha\| < m_r$ and $x_i \in I$, then consider all the pairs of i-neighboring spheres, which are included into subcube $C(\alpha, 1)$. Let (S_1, S_2) be such a pair and these spheres are in $(n - m_r)$-subcubes D_1 and D_2 respectively. Then replace S_1 to S_2 in subcube D_1 and replace S_2 to S_1 in D_2. Proceed analogously for all other pairs of i-neighboring spheres;

2). If $\|\alpha\| = m_r$ and $x_i \notin I$, then invert the i-th entry of all points of the set $U_{m_r}(A, I)$;

3). Otherwise the transformation is undefined.

We say that points $\alpha_1, ..., \alpha_q$ and coordinates $x_1, ..., x_q$ of B^n satisfy the condition W if

1). $1 \leq \|\alpha_1\| < \|\alpha_2\| < \cdots < \|\alpha_q\| \leq m_r$;

2). $C(\alpha_1, 1) \supseteq C(\alpha_2, 1) \supseteq \cdots \supseteq C(\alpha_q, 1)$;

3). $\{x_1, ..., x_q\} \subseteq I$ if $\|\alpha_q\| < m_r$ and $\{x_1, ..., x_{q-1}\} \subseteq I$, $x_q \notin I$ if $\|\alpha_q\| = m_r$.

Theorem 1 *For any $A \in \mathcal{K}(n,m) \backslash \mathcal{L}(n,m)$ there exist points $\gamma_1, ..., \gamma_p$ and coordinates $y_1, ..., y_p$, satisfying the condition W, such that*

$$A = R(\gamma_p, y_p) R(\gamma_{p-1}, y_{p-1}) ... R(\gamma_1, y_1) B$$

for some $B \in \mathcal{L}(n,m)$.

Now we are going to show a way how to determine the points $\gamma_1, ..., \gamma_p$ and coordinates $y_1, ..., y_p$ for a set $A \in \mathcal{K}(n,m) \backslash \mathcal{L}(n,m)$. Notice that if a set $U_{m_r}(A, I)$ is not i-normalized then such coordinate x_i is unique and is the minimal coordinate for it. Denote by $N(A, I)$ the collection of all indices $a_1, ..., a_t$, such that $U_{a_i}(A, I)$ is not normalized, and let $N(A) = \cap_{i \in I} N(A, I)$.

Theorem 2 *For any set $A \in \mathcal{K}(n,m) \backslash \mathcal{L}(n,m)$ there exists a sequence $I_A = \{x_{i_1}, ..., x_{i_m}\}$ (with $m = m_r$), such that*

(i) $N(A, I_A) = N(A) = \{a_1, ..., a_t\}$;

(ii) the set $U_{a_j}(A, I_A)$, $1 \leq j \leq t$, is not a_j-normalized, and if $a_j < m_r$, then $a_j = x_{i_{j+1}}$ holds.

Consider now a sequence $I_A = \{x_{i_1}, ..., x_{i_m}\}$ (with $m = m_r$), and the set $N(A) = \{j_1, ..., j_r\}$. Assume $j_1 < j_2 < \cdots < j_r$. Consider vector $\kappa = (\kappa_1, ..., \kappa_n) \in B^n$, such that $\kappa_l = 0$ iff $l \notin \{i_1, ..., i_m\}$ $(m = m_r)$, or if $l = i_s$ for some s, $1 \leq s \leq m_r$, and in the canonical representation of δ there is a binomial coefficient of the form for some c. Then the i-th entry of γ_q, $q = 1, ..., t$, equals κ_i if $i < j_q$ and 0 otherwise.

References

[1] Aslanjan L.A. *The isoperimetric problem and related problems for discrete spaces* (in Russian), Problemy Kibernetiky **36** (1979), 85–127.

[2] Harper L.H. *Optimal numberings and isoperimetric problems for graphs*, J. Combin. Theory **1**(1966), No 3, 385–393.