Embedding Ladders and Caterpillars into the Hypercube

S.L. Bezrukov [1], B. Monien [1], W. Unger [1]

Fachbereich Mathematik/Informatik, Universität–GH Paderborn, Fürstenallee 11, D–33102 Paderborn, Germany

G. Wechsung

Institut für Theoretische und Praktische Informatik, Friedrich–Schiller–Universität Jena, Leutragraben 1, D–07743 Jena, Germany

Abstract

We present embeddings of generalized ladders as subgraphs into the hypercube. By embedding caterpillars into ladders, we obtain embeddings of caterpillars into the hypercube. In this way we obtain almost all known results concerning the embeddings of caterpillars into the hypercube. In addition we construct embeddings for some new types of caterpillars.

Key words: Embedding, caterpillars, hypercube.

1 Introduction

Embedding of graphs is an important and well-studied theory in parallel computing. Much research has been devoted to finding "good" embeddings of one processor network into another. See [18] for an overview on embeddings. We study embeddings of ladders and caterpillars into the hypercube. Before stating the known results on this topic we give the formal definition of an embedding. All the graph-theoretical concepts which are not defined here can be found in any introductory book on graph theory (e.g. [7]).

[1] Supported by the DFG-Sonderforschungsbereich 376 "Massive Parallelität: Algorithmen, Entwurfsmethoden, Anwendungen" and the EC ESPRIT Long Term Research Project 20244 (ALCOM-IT).

Definition 1 *The embedding of a guest graph $G = (V, E)$ into a host graph $H = (V', E')$ is defined by an injective function $f : V \mapsto V'$. If $(u, v) \in E$, then we call the distance between $f(u)$ and $f(v)$ in H the dilation of the edge (u, v). The maximal dilation over all edges of G is called the dilation of the embedding f. The expansion of the embedding is the ratio $|V'|/|V|$.*

Embedding G into H with minimal dilation and expansion is important for network design and for the simulation of one computer architecture by another. The problem of verifying whether G is embeddable into H with dilation d is NP-complete for general graphs [5]. It remains NP-complete when the guest graph G is a tree and H is a hypercube [19].

Definition 2 *Denote by Q^n the n-dimensional hypercube. The vertex set of this graph is the collection of all binary strings of length n. Two vertices of Q^n are adjacent iff the corresponding strings differ in exactly one entry.*

Let G be a graph with m vertices. The hypercube of dimension $\lceil \log_2(m) \rceil$ is called its optimal hypercube, and one of dimension $\lceil \log_2(m) \rceil + 1$ is called the next-to-optimal.

In [2] trees of degree bounded with a constant and their embeddings into the hypercube are considered. It is shown that there exist embeddings with dilation also bounded with a constant. A conjecture of Havel [12] claims that trees of maximal degree 3 with 2^n vertices can be embedded into their optimal hypercubes with dilation at most 2. In [17] two embeddings of trees of degree 3 are constructed, one with dilation 5 into the optimal hypercube and one with dilation 3 and constant expansion.

Definition 3 *A bipartite graph is called balanced, if there is a vertex two-coloring, where the two color sets have the same number of vertices.*

Since a hypercube itself is balanced, then each of its spanning subgraphs has to be balanced. Another conjecture of Havel [11] states that any balanced tree of degree 3 is a subgraph of its optimal hypercube.

The results above show that the embedding of trees into Q^n is a difficult problem which is far from being completely solved. One possible direction of research is to consider some special classes of trees, e.g. caterpillars.

Definition 4 *A caterpillar C is a tree of maximal degree 3 where there exists a path B (called the backbone of C) so that, after deleting all edges of B, C consists of a set of paths. These paths are called the legs of C.*

Caterpillars and their embeddings were investigated in a number of papers, the results of which show that embedding problems are nontrivial even in this case. In [6,16] the authors studied the complexity aspects of the problem of

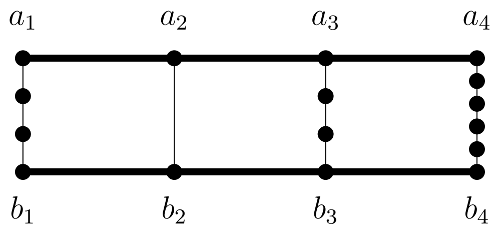


Fig. 1. A ladder with 4 rungs consisting of 4, 2, 4 and 6 vertices

determining the minimal dilation of embedding a caterpillar into a path, and proved its NP-completeness. In the case of the hypercube, the embedding of caterpillars is relatively well-studied. Following the conjecture of Havel [11], each balanced caterpillar is embeddable with dilation 1 (i.e. as a subgraph) into its optimal hypercube. There is a number of papers devoted to the proof of this conjecture for particular types of caterpillars (see [11–14]), but the general case remains open.

In Section 2 we introduce (generalized) ladders and show that each such ladder is a subgraph of its optimal hypercube. This result is used in Section 3 for embedding caterpillars. It turned out that most types of caterpillars previously considered in relevant literature are subgraphs of the generalized ladders. Thus, the approach to embedding based on ladders implies many known results on embeddings of caterpillars and also provides embeddings for many new types of caterpillars. Section 4 concludes the paper. There we show that two copies of any caterpillar C form a subgraph of the next-to-optimal (with respect to C) hypercube and consider generalized caterpillars.

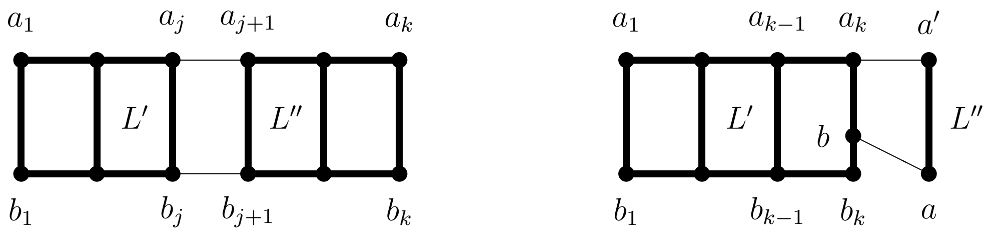
2 Ladders and their embedding

Definition 5 *Consider two paths $[a_1, ..., a_k]$ and $[b_1, ..., b_k]$ and join each pair of vertices a_i, b_i, $i = 1, ..., k$, with a new path. The resulting graph is called a ladder, and the paths between a_i, b_i are called its rungs (cf Fig. 1).*

Throughout this paper we assume that each rung of the ladder consists of an even number of vertices, so that each of its cycles has even length. This is, of course, necessary for showing that a ladder is a subgraph of the hypercube.

Theorem 6 *Any ladder is a subgraph of its optimal hypercube.*

PROOF. Let L be a ladder with k rungs $r_1, ..., r_k$ and n be the dimension of its optimal hypercube. It is sufficient to consider the case $|V(L)| = 2^n$. Indeed,



a. b.

Fig. 2. Cases 1 and 2 of the proof of Theorem 1

if $|V(L)| < 2^n$, we add to L one more rung with $2^n - |V(L)|$ vertices and get a ladder $\tilde{L}$. Now the validity of Theorem 6 for $\tilde{L}$ would imply its validity for L.

Since Q^n is a hamiltonian graph, we assume without loss of generality that $k > 1$. Denote by a_i, b_i $(i = 1, ..., k)$ the terminal vertices of the rung r_i with a_i adjacent to a_{i+1} and b_i adjacent to b_{i+1} for $i = 1, ..., k-1$ (cf. Fig. 1).

We proceed by induction on n, assuming that for any $\tilde{n} < n$ and any ladder $\tilde{L}$ with $\tilde{k} \geq 1$ rungs and $2^{\tilde{n}}$ vertices there exists an embedding $\tilde{\phi}$ of $\tilde{L}$ into $Q^{\tilde{n}}$ with dilation 1 so that $(\tilde{\phi}(a_1), \tilde{\phi}(b_1)), (\tilde{\phi}(a_{\tilde{k}}), \tilde{\phi}(b_{\tilde{k}})) \in E(Q^{\tilde{n}})$.

For $n = 2$ the validity of the inductive hypothesis is easily verified, so let $n \geq 3$. Denote by m_i the number of vertices in the rung r_i, $i = 1, ..., k$.

Case 1 *Let* $\sum_{i=1}^{j} m_i = 2^{n-1}$ *for some* j, $1 < j < k$.

We split L into two ladders L' and L'' with 2^{n-1} vertices each. The ladder L' consists of the first j rungs of L and L'' consists of the remaining $k - j$ rungs (cf. Fig. 2a).

Now partition Q^n into two hypercubes Q' and Q'' of dimension $n-1$. Embed L' as a subgraph into Q' and embed L'' as a subgraph into Q'' by induction. Let $x', y' \in V(Q')$ be the images of a_j, b_j and let $x'', y'' \in V(Q'')$ be the images of a_{j+1}, b_{j+1} in the corresponding embeddings respectively.

Denote by u, v the vertices of Q'' such that (x', u) and (y', v) are edges of Q^n. Since $(x', y'), (u, v) \in E(Q^n)$, there exists an automorphism ψ of Q'' so that $\psi(x'') = u$ and $\psi(y'') = v$. In other words, there exists an embedding of L'' as a subgraph into Q'' so that $x'' = u$ and $y'' = v$. Adding the edges (x', u) and (y', v), we get an embedding ϕ of the whole ladder L as a subgraph into Q^n, for which $(\phi(a_1), \phi(b_1)), (\phi(a_k), \phi(b_k)) \in E(Q^n)$ holds.

Case 2 *Let* $m_k > 2^{n-1}$.

There exist edges (a, b) and (a_k, a') in r_k so that the rung r_k may be cut into three paths: $r_1' = [b_k, b]$, $r_2' = \{a_k\}$ and $r'' = [a, a']$ with $m_k - 2^{n-1} - 1$, 1

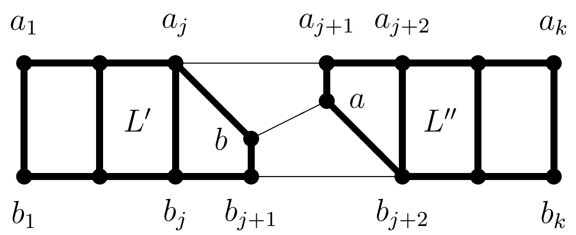


Fig. 3. Case 4 of the proof of Theorem 1

and 2^{n-1} vertices respectively (if $m_k = 2^{n-1} + 2$ then $b = b_k$). By adding the edge (a_k, b) the ladder L', consisting of the rungs $r_1, ..., r_{k-1}, r_1' \cup r_2'$, is formed. Denote by L'' the ladder, consisting of the single rung r'' (see Fig. 2b). As in Case 1, the ladders L' and L'' are embedded into the hypercubes Q' and Q'' of dimension $n-1$ by induction. Then, the embedding of L'' is changed by an automorphism of Q'' so that the images of a_k and a' and those of b and a in the resulting embedding are adjacent in Q^n.

Case 3 *Let* $m_1 > 2^{n-1}$.

In this case we proceed in a similar fashion as in Case 2.

Case 4 *Let* $\sum_{i=1}^{j} m_i + \delta = 2^{n-1}$ *for some* j, $1 < j < k-1$ *and* δ, $0 < \delta < m_{j+1}$.

Note that δ is even and consider the rung r_{j+1} of L. By cutting an appropriate edge (a, b) of r_{j+1}, split it into two paths $r' = [b, b_{j+1}]$ and $r'' = [a_{j+1}, a]$ with δ and $m_{j+1} - \delta$ vertices respectively (see Fig. 3). Denote by L' the ladder formed by the rungs $r_1, ..., r_j, r'$ and the edge (a_j, b) and denote by L'' the ladder formed by the rungs $r'', r_{j+2}, ..., r_k$ and the edge (a, b_{j+2}). The ladders L', L'' are shown by bold lines in Fig. 3 and have 2^{n-1} vertices each.

Now, once more partition Q^n into two hypercubes Q' and Q'' of dimension $n-1$. Embed L' as a subgraph into Q' and embed L'' as a subgraph into Q'' by induction. Let $x', y', z' \in V(Q')$ be the images of a_j, b, b_{j+1} respectively in the embedding of L' and let $x'', y'', z'' \in V(Q'')$ be the images of a_{j+1}, a, b_{j+2} respectively in the embedding of L''.

Denote by u, v, w the vertices of Q'' with $(x', u), (y', v), (z', w) \in E(Q^n)$. By inductive hypothesis the vertices x', y', z' and x'', y'', z'' form paths of length 2 in Q' and Q'' respectively. Since the vertices u, v, w also form a path of length 2 in Q'', there exists an automorphism ψ of Q'' so that $\psi(x'') = u$, $\psi(y'') = v$ and $\psi(z'') = w$. In other words, there exists an embedding of L'' as a subgraph into Q'' so that $x'' = u$, $y'' = v$, and $z'' = w$. By adding the edges (x', u), (y', v) and (z', w) we obtain an embedding ϕ of the whole ladder L into Q^n as a subgraph, completing the induction. $\square$

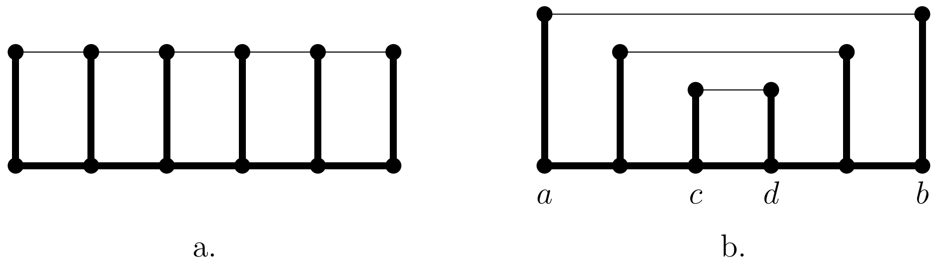


Fig. 4. Embedding some caterpillars into ladders

3 Application of ladders for embedding caterpillars

We recall the conjecture of Havel [11] that each balanced tree with maximal degree at most 3 and 2^n vertices is a subgraph of Q^n.

This conjecture still remains open, whereas for certain types of caterpillars it is known to be true. In particular it is proved for caterpillars with all legs of length 0 or 1 [14] and for caterpillars with lengths of all legs of the same parity [13]. In [12], one more class of caterpillars is introduced and some partial results are obtained. Here, we show that embedding of almost all types of caterpillars listed above can easily be obtained by embedding them into ladders and then using Theorem 1. Furthermore, we solve a problem stated in [12] and present a wide class of caterpillars embeddable as subgraphs into Q^n.

Corollary 7 *Let C be a caterpillar, each leg of which has an even number of vertices. Then C is a subgraph of its optimal hypercube.*

PROOF. Join by edges the second ends of the legs, which are vertices of degree 1 (cf. Fig. 4a). We get a ladder and apply Theorem 6. □

In [12] the following family $\{A_m\}$ of caterpillars is considered (see Fig. 5). The caterpillar A_m has m legs with $1, 2, ..., m$ vertices. The paper [12] provides a nice way for embedding such caterpillars into the optimal hypercube, which works only when m is of the form $m = 2^p - 1$ for some p. In general, however, (e.g. for $m = 5$) the caterpillar A_m is not a subgraph of its optimal hypercube. The next theorem determines the minimal dimension of the hypercube containing A_m as a subgraph, which answers the corresponding question in [12].

Proposition 8 *A_m is a subgraph of Q^n with $n = \left\lceil \log_2 \left(\frac{m(m+1)}{2} + \lceil \frac{m}{2} \rceil \right) \right\rceil$, and this n is the minimal possible.*

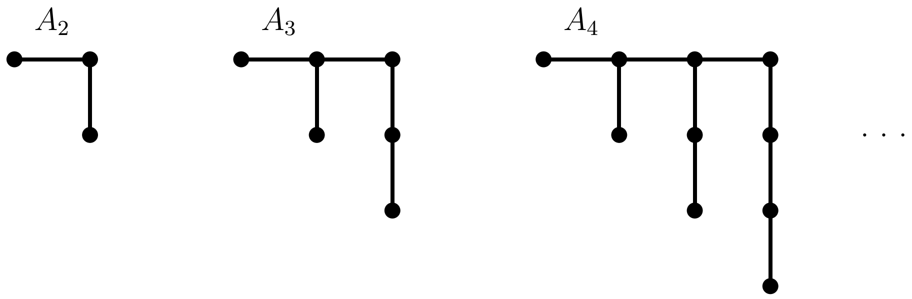


Fig. 5. The family A_m of caterpillars

PROOF. Denote by n the minimal dimension of the hypercube, containing A_m as a subgraph. Let us add one extra vertex to each leg of A_m with an odd number of vertices and thus get a caterpillar C with an even number of vertices in each leg, which one can embed into its optimal hypercube by Corollary 7. The only problem is that we may have added too many vertices and this made C a subgraph of $Q^{n'}$ with $n' > n$. In the following, we show that $n' = n$.

For this, we consider a 2-coloring of A_{2k}, where the color classes are always of size k^2 and k^2+k. The difference between these two numbers is k, which equals the number of the odd legs of A_{2k}. Thus, if we embed A_{2k} into the hypercube Q^n of minimal dimension, at least k vertices of Q^n are free, because the color classes of Q^n in any 2-coloring are of size 2^{n-1}. Thus, the dimension of the optimal hypercube for the extended caterpillar C is n.

Similarly for A_{2k+1} the color classes are of size k^2+k and $(k+1)^2$. The difference between these numbers again equals the number of odd legs of A_{2k+1}.

Therefore, if A_m is a subgraph of Q^n, then $n \geq \lceil \log_2(|V(C)|) \rceil$. On the other hand, A_m is a subgraph of C and C is a subgraph of its optimal hypercube. $\square$

Now, we present a new type of caterpillars which are subgraphs of their optimal hypercubes. Note that the backbone of a caterpillar is not determined uniquely. It is easily shown that if a caterpillar is balanced, one can always find a backbone consisting of an even number of vertices. We denote this number by $2b$ and denote the vertices of the backbone by $v_1, v_2, ..., v_{2b}$, assuming that they are labeled consecutively from one of the caterpillar's ends to the other. Let l_i be the number of vertices in the leg incident with the vertex v_i (counted together with the vertex in the backbone) and let w_i be the other endpoint of this leg, $i = 1, ..., 2b$.

Theorem 9 *Let C be a balanced caterpillar so that $l_i + l_{2b-i+1}$ is even for $i = 1, 2, ..., b$. Then C is a subgraph of its optimal hypercube.*

PROOF. Denote the endpoints of the backbone by a, b. Then there exist adjacent vertices c, d in the backbone (assume c is between a and d) so that the paths a, c and d, b are of the same length, i.e. (c, d) is the central edge of the backbone (see Fig. 4b). Adding to C the edges of the form (w_i, w_{2b-i+1}), we transform C into a ladder. Now the theorem follows from Theorem 6. □

This theorem works for many caterpillars, in particular for the one considered in [13]:

Corollary 10 *Let C be a balanced caterpillar, each leg of which has an odd number of vertices. Then C is a subgraph of its optimal hypercube.*

PROOF. Clearly, the number of vertices in the backbone of C has to be even and the assertion follows from Theorem 9. □

4 Conclusion

Clearly, if a ladder is a subgraph of the hypercube then the number of vertices in each its rung is of the same parity (otherwise the ladder contains a cycle of an odd length). It is interesting to mention that the balanced ladder with 32 vertices, which has rungs consisting of 3, 3, 3, 3, 3, 3, 3, 3, 5, 3 vertices (counted from one end of the ladder to the other) is not a subgraph of Q^5 [3].

Concerning the caterpillars, if a caterpillar C is a subgraph of its optimal hypercube Q^n, then, obviously, two copies of C form a subgraph of Q^{n+1}. The next proposition may be considered as a support of Havel's conjecture [11] on caterpillars. We embed two copies of C into the next-to-optimal hypercube without embedding C into its optimal hypercube.

Proposition 11 *Let C be an arbitrary caterpillar. Then:*

a. *One can embed 2 copies of C into a hypercube which is the next-to-optimal with respect to C with dilation 1;*
b. *C is a subgraph of its next-to-optimal hypercube.*

PROOF. To prove the first assertion, form a ladder from the two copies of C, placing them symmetrically and joining by edges the ends of the corresponding legs (which are vertices of degree 1). The rungs of such a ladder always consist of an even number of vertices. The second assertion follows from the first one. □

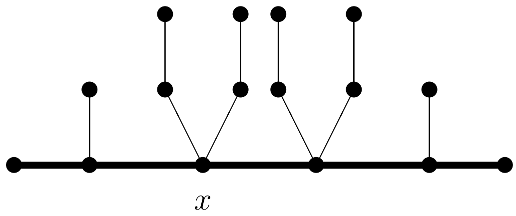


Fig. 6. A caterpillar of order 2

Note that Proposition 11a holds for the balanced caterpillars as well as for non-balanced ones. Moreover, if a caterpillar is not balanced, it cannot be a subgraph of its optimal hypercube in general. Thus, the upper bound in Proposition 11b cannot be improved.

Proposition 12 (see [4]). *Any caterpillar is embeddable into its optimal hypercube with dilation 2.*

This result can be also considered as a support to the conjecture of Havel [12] concerning embedding of binary trees into their optimal hypercube.

Definition 13 *A caterpillar of order k ($k \geq 1$) is a tree of maximal degree $k+2$, such that all the vertices of degree* 3 *and greater belong to a single path.*

Only few results are known about caterpillars of higher orders $k \geq 2$. These include the results of [4,8,9] concerning embedding of a particular type of caterpillars of orders 2 and 3. In [10,15] it is shown by various methods that the number of caterpillars with p vertices ($p \geq 5$) equals $2^{p-4} + 2^{\lfloor (p-4)/2 \rfloor}$.

In [1] we present another approach to embedding of ladders and caterpillars which works, in particular, for more types of caterpillars of order 1. It is interesting that for caterpillars of order 2 similar propositions hold:

Theorem 14 (see [1]). *Let C be a caterpillar of order* 2*. Then:*

a. C is a subgraph of its next-to-optimal hypercube.
b. C is embeddable into its optimal hypercube with dilation 2.

However, the upper bound in Theorem 14a is attainable on non-balanced caterpillars and on some balanced ones as well. The caterpillar shown in Fig. 6 is balanced, has 16 vertices, but is not a subgraph of Q^4. The number of vertices at distance 2 from the vertex x is 7, whereas it has to be at most 6 in a subgraph of Q^4. Finally, Theorem 14b implies, in particular, Proposition 12.

Acknowledgement

The authors would like to thank the referees for their remarks that improved the presentation.

References

[1] S.L. Bezrukov, B. Monien, W. Unger and G. Wechsung, *Dense subsets and their use in embedding problems* (Preprint, 1995).

[2] S.N. Bhatt, F.R.K. Chung, F.T. Leighton and A.L. Rosenberg, Efficient Embedding of Trees in Hypercube, *SIAM J. Comput.* **21** (1990) No.1 151–162.

[3] R. Caha and V. Koubek, (private communication, April 1996).

[4] T. Dvořák, I. Havel, J.-M. Laborde and M. Mollard, *Spanning Caterpillars of the Hypercube* (Technical Report RR 944.M, LSD Grenoble, 1995).

[5] M.R. Garey and D.S. Johnson, *Computers and Intractability: a Guide to the Theory of NP-Completeness* (Freeman, 1979).

[6] J. Haralambides, R. Makedon and B. Monien, Bandwidth Minimization: An Approximation Algorithm for Caterpillars, *Mathematical Systems Theory* **24** (1991) 169–177.

[7] F. Harary, *Graph Theory* (Addison-Wesley Publ. Company, 1969).

[8] F. Harary and M. Lewinter, Spanning Subgraphs of a Hypercube VI: Survey and Unsolved Problems, in: *Graph theory combinatorics and applications, Vol. 2* (Kalamazoo, MI, 1988) 633–637.

[9] F. Harary, M. Lewinter and W. Widulski, On Two-legged Caterpillars which Span Hypercubes, *Congressus Numeratium* **66** (1988) 103–108.

[10] F. Harary and A.J. Schwenk, The Number of Caterpillars, *Discr. Math.* **6** (1973) 359–365.

[11] I. Havel, On Hamiltonian Circles and Spanning Trees of Hypercube Graphs, *Čas. Pést. Mat.* **109** (1984) 135–152.

[12] I. Havel, On Certain Trees in Hypercube, in: R. Bodendiek and R. Henn, eds., *Topics in Combinatorics and Graph Theory* (Physica-Verlag, Heidelberg 1990) 353–358.

[13] I. Havel, J.-M. Laborde and P. Liebl, *Caterpillar-like Trees Spanning Hypercubes*, (Preprint).

[14] I. Havel and P. Liebl, One-legged Caterpillars Span Hypercubes, *J. Graph Theory* **10** (1986) 69–78.

[15] A. Lifschitz and A. Redstone, A Determination of the Number of Caterpillars, *Southeast Asian Bull. Math.* **9** (1985) No.1 50–52.

[16] B. Monien, The Bandwidth Minimization Problem for Caterpillars with Hair Length 3 is NP-complete, *SIAM J. Algebraic and Discrete Methods* **7** (1986) No.4 505–512.

[17] B. Monien and I.H. Sudborough, Simulating Binary Trees on Hypercubes, in: *Proceedings Third Aegan Workshop on Computing, Lecture Notes Computer Science* **319** (Springer, 1988) 170–180.

[18] B. Monien and I.H. Sudborough, Embedding one Interconnection Network in Another, *Computing Suppl.* **7** (1990) 257–282.

[19] A.S. Wagner, *Embedding Trees into the Hypercube* (Technical Report 204/87, Dept. of Computer Science, University of Toronto, October 1987).