On Posets whose Products are Macaulay

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Abstract

If P is an upper semilattice whose Hasse diagram is a tree and whose cartesian powers are Macaulay, it is shown that Hasse diagram of P is actually a spider in which all the legs have the same length.

1 Introduction

Let $(P, \subseteq)$ be a finite poset. For $x, y \in P$ we write $x \subset\cdot\, y$ if $x \subset y$ and there is no $z \in P$ yielding $x \subset z \subset y$. The poset $(P, \subseteq)$ is called *ranked* if there exists a function $r_P : P \mapsto \mathbb{N}$ such that $\min_{x \in P} r_P(x) = 0$ and for any $x, y \in P$ the condition $x \subset\cdot\, y$ implies $r_P(x) + 1 = r_P(y)$. We call the numbers $r_P(x)$ and $r_P = \max_{x \in P} r_P(x)$ the *rank* of x and P respectively. The set

$$P_i = \{x \in P \mid r_P(x) = i\}$$

is called the i^{th} *level* of P. For a subset $A \subseteq P_i$ and $i > 0$ define the *shadow* of A as

$$\Delta(A) = \{x \in P_{i-1} \mid x \subset\cdot\, y \text{ for some } y \in A\}.$$

The *the shadow minimization problem* plays an important role in combinatorics and is often in the background of various extremal problems: for a given poset $(P, \subseteq)$ and given natural numbers $i > 0$ and m, $1 \leq m \leq |P_i|$, find a subset $A \subseteq P_i$ such that $|A| = m$ and $|\Delta(A)| \leq |\Delta(B)|$ for any $B \subseteq P_i$ with $|B| = m$. We are interested in the case when the shadow minimization problem has a *nested structure of solutions*, which leads to the notion of a *Macaulay poset*.

Let $\preceq$ be a total order on P. For $z \in P_i$ denote

$$\mathcal{F}_i(z) = \{x \in P_i \mid x \preceq z\}.$$

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We call a subset $A \subseteq P_i$ *initial segment* if $A = \mathcal{F}_i(z)$ for some $z \in P_i$. A poset $(P, \subseteq)$ is called *Macaulay*, if there exists a total order $\preceq$ (called *Macaulay order*), such that the following properties hold:

N$_1$ (*nestedness*) : For any $z \in |P_i|$, and any $i > 0$ the initial segment $\mathcal{F}_i(z)$ has minimal shadow among all subsets of P_i of the same cardinality;

N$_2$ (*continuity*) : For $i > 0$ it holds: $\Delta(\mathcal{F}_i(z)) = \mathcal{F}_{i-1}(z')$ for some $z' \in P_{i-1}$.

We concentrate our attention on posets which are factorable by using the cartesian product operation. The *cartesian product* of two posets $(P, \subseteq_P)$ and $(Q, \subseteq_Q)$ is a poset with the element set $P \times Q$ and the partial order $\subseteq_\times$ defined as follows. We say $(x', y') \subseteq_\times (x'', y'')$ iff $x' \subseteq_P x''$ and $y' \subseteq_Q y''$. Clearly, if P and Q are ranked posets, then $P \times Q$ is a ranked poset as well, where $r_{P\times Q}(x, y) = r_P(x) + r_Q(y)$. Since the cartesian product is an associative operation, the product of more than two posets is well defined. We denote by $(P^n, \subseteq_\times)$ the n^{th} cartesian power of a poset $(P, \subseteq)$.

The shadow minimization problem for cartesian powers of various posets was considered in a number of papers. We refer to the book [4] for an excellent overview on the subject. Presently, just for the posets shown in Fig. 1a-c it is known that any of their cartesian powers is a Macaulay poset (cf. [5, 6], [3] and [1, 7, 8, 9] respectively).

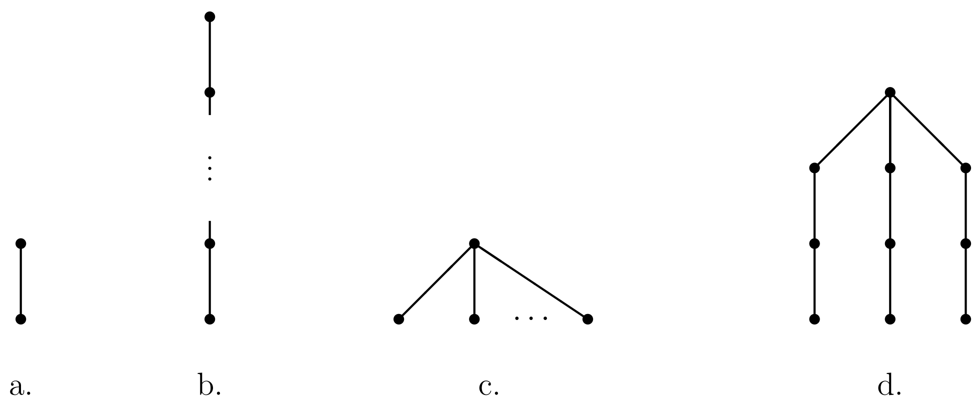

Figure 1: The basic posets

These cartesian powers of the posets shown in Fig. 1a-c are classical posets in combinatorics and are known as the *Boolean poset*, the *lattice of multisets*, and the *star poset* respectively. Evidently, all these posets have something in common, namely, their Hasse diagrams are trees. They are also *upper-semilattices*. In any poset $(P, \subseteq)$, for $a, b \in P$, $\sup_P(a, b)$ denotes an element $c \in P$ (if it exists) such that $a \prec c$, $b \prec c$ and $c \prec d$ if $a \prec d$ and $b \prec d$. A poset $(P, \subseteq)$ is an *upper-semilattice* if for any $a, b \in P$, $\sup_P(a, b)$ exists and is unique.

Denote by $\mathcal{P}$ the class of upper semilattices P whose Hasse diagrams are trees. In this paper we will show that if $P \in \mathcal{P}$ and P^n is Macaulay for some integer $n \geq r_P + 3$, then the Hasse diagram of P is a regular *spider* with all legs having the same length (cf. Fig. 1d). In [2] we will prove that if the Hasse diagram of P is a regular spider, then its products P^n, $n = 1, 2, \ldots$ are Macaulay.

2 Some properties of Macaulay posets

Let $(P, \subseteq)$ be a Macaulay poset, $A \subseteq P_i$ with $|A| = m$. For $2 \leq l \leq i-1$ denote

$$\Delta_l(A) = \Delta\left(\Delta_{l-1}(A)\right), \quad \text{with } \Delta_1(A) = \Delta(A).$$

Thus, $\Delta_l(A) \subseteq P_{i-l}$. The following lemma can be easily proved by induction on l.

Lemma 1 (cf. [1, 4]) *Let $(P, \subseteq)$ be a Macaulay poset. Then for any $z \in P_i$ it holds: $|\Delta_l(\mathcal{F}_i(z))| \leq |\Delta_l(A)|$ for any $A \subseteq P_i$ with $|A| = |\mathcal{F}_i(z)|$. Moreover, $\Delta_l(\mathcal{F}_i(z)) = \mathcal{F}_{i-l}(z')$ for some $z' \in P_{i-l}$.*

Now, assuming $i < r_P$, we introduce the *upper shadow* of the set $A \subseteq P_i$ defined by

$$\nabla(A) = \{x \in P_{i+1} \mid \exists y \in A \text{ with } y \subset\cdot\, x\}.$$

For a total order $\preceq$ and $z \in P_i$ denote $\mathcal{L}_i(z) = \{x \in P_i \mid z \preceq x\}$ and call such a set *final segment*.

Lemma 2 (cf. [1, 4]) *Let $(P, \subseteq)$ be a Macaulay poset. Then for any $z \in P_i$ it holds: $|\nabla(\mathcal{L}_i(z))| \leq |\nabla(A)|$ for any $A \subseteq P_i$ with $|A| = |\mathcal{L}_i(z)|$. Moreover, $\nabla(\mathcal{L}_i(z)) = \mathcal{L}_{i+1}(z')$ for some $z' \in P_{i+1}$.*

3 Macaulay posets and the class $\mathcal{P}$

Throughout this section we denote the elements of P by Greek letters and represent the elements of P^n by n-dimensional vectors denoted by bold Latin letters.

Denote by $Q(k, l) \in \mathcal{P}$ the poset, whose Hasse diagram is obtained from k disjoint chains of length l each by identifying their top vertices. The graph of the Hasse diagram of $Q(k, l)$ is a spider with k legs consisting of l vertices each. The example of $Q(3, 3)$ is shown in Fig. 1d.

The main result of the paper is the following theorem.

Theorem 1 *Suppose for some poset $(P, \subseteq) \in \mathcal{P}$ that $(P^n, \subseteq_\times)$ is Macaulay for some integer $n \geq r_P + 3$. Then $(P, \subseteq) = Q(k, l)$ for some $k \geq 1$ and $l \geq 1$.*

In order to prove this theorem we need some auxiliary propositions. For $\alpha, \beta \in P$ with $\alpha \subseteq \beta$ introduce the intervals

$$\begin{aligned} I(\alpha,\beta) &= \{\gamma \in P \mid \alpha \subseteq \gamma \subseteq \beta\}, \\ I(\beta) &= \{\gamma \in P \mid \gamma \subseteq \beta\}, \\ I_i(\beta) &= I(\beta) \cap P_i. \end{aligned}$$

Denote by U_P the universal upper bound of a poset $(P, \subseteq) \in \mathcal{P}$, i.e. the element of P, such that $\alpha \subseteq U_P$ for any $\alpha \in P$. We call a vertex of a tree *leaf* if it is incident with exactly one edge of the tree.

Lemma 3 *With the assumptions of Theorem 1, for any leaf α of the Hasse diagram of the poset $(P, \subseteq)$ it holds: $r_P(\alpha) \in \{0, r_P\}$.*

Proof.
Note that any tree has at least two leaves and that all elements of P_0 are leaves. Therefore, if the Hasse diagram of P has exactly two leaves and one of them is U_P, then $(P, \subseteq) = Q(1,l)$ for some $l \geq 1$ and the lemma is true. Furthermore, if all the leaves of the Hasse diagram have the same rank, then the validity of the lemma follows from the definition of the class $\mathcal{P}$.

It remains to show that if β is a leaf and $\beta \neq U_P$, then $r_P(\beta) = r > 0$ is impossible. Let α be a leaf of rank 0, let $\gamma = \sup_P(\alpha, \beta)$ and let $r_P(\gamma) = t$ (cf. Fig. 2a).

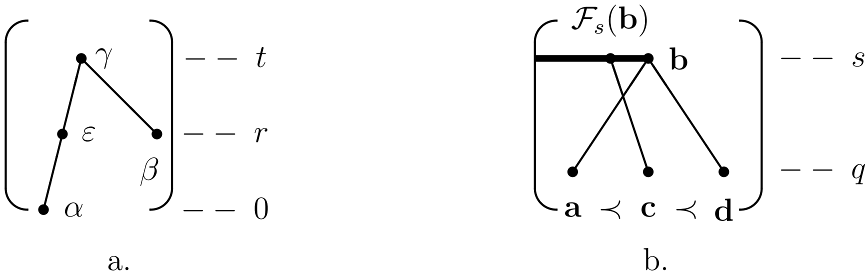

Figure 2: Fragments of P (a.) and P^n (b.) used in the proof of Lemma 3

With $n \geq r_P + 3$, $s = t(n-1) - 1$ and $q = r(n-1)$, we have $s > q > 0$. Now consider the set

$$M = \{(\xi_1, \ldots, \xi_n) \in P^n_q \mid \xi_i \in \{\alpha, \beta\}, \quad i = 1, \ldots, n\},$$

and let $\mathbf{a}$ be the first vector in M (in the Macaulay order $\preceq$). Thus, some $(n-1)$ entries of $\mathbf{a}$ are β and the remaining entry is α. We may assume without loss of generality that

$$\mathbf{a} = (\alpha, \beta, \beta, \ldots, \beta).$$

Let $\mathbf{b} = (\delta_1, \ldots, \delta_n)$ be the first vector of P^n_s such that $\mathbf{a} \in \Delta_{s-q}(\mathcal{F}_s(\mathbf{b}))$. There are at least two entries δ_i, δ_j of $\mathbf{b}$ such that $r_P(\delta_i) \geq t$ and $r_P(\delta_j) \geq t$, since we would otherwise have $r_{P^n}(\mathbf{b}) \le (n-1)(t-1) + r_P = t(n-1) - 2 < s$. We may, therefore, assume without loss of generality that $r_P(\delta_2) \ge t$. Since $\beta \subset \delta_2$ and $r_P(\gamma) \le r_P(\delta_2)$, it follows that $\gamma \subseteq \delta_2$.

Similarly there exist at least two entries δ_i, δ_j of $\mathbf{b}$ such that $r_P(\delta_i) < t$ and $r_P(\delta_j) < t$, since we would otherwise have $r_{P^n}(\mathbf{b}) \ge t(n-1) > s$. We assume, without loss of generality, that $r_P(\delta_3) < t$, so $\delta_3 \supseteq \beta$ (since $\mathbf{b} \supseteq \mathbf{a}$) and $\delta_3 \ne \gamma$. Denote by ε the element in $I(\alpha, \gamma)$ with $r_P(\varepsilon) = r_P(\beta)$ (cf. Fig. 2a). Such an element exists and is unique. Further consider the elements $\mathbf{c}$ and $\mathbf{d}$ of P_q^n defined by

$$\mathbf{c} = (\beta, \beta, \alpha, \beta, \ldots, \beta), \quad \mathbf{d} = (\alpha, \varepsilon, \beta, \beta, \ldots, \beta).$$

Since α and β are leaves, $\Delta(\mathbf{a}) = \Delta(\mathbf{c}) = \emptyset$. It follows that $\Delta(\mathcal{F}_q(\mathbf{c})) = \emptyset$, for if $\mathbf{f}$ is the first element in $\mathcal{F}_q(\mathbf{c})$ with $\Delta(\mathbf{f}) \ne \emptyset$, then $\mathbf{f} \prec \mathbf{c}$ and

$$|\Delta(\mathcal{F}_q(\mathbf{f}))| > |\Delta((\mathcal{F}_q(\mathbf{f}) \setminus \mathbf{f}) \cup \mathbf{c})| = 0,$$

contradicting N_1. Thus $\Delta(\mathcal{F}_q(\mathbf{c})) = \emptyset$ is established.

Since $\Delta(\mathbf{d}) \ne \emptyset$, $\mathbf{c} \prec \mathbf{d}$ follows. Since $\mathbf{a}$ is the first vector in M, since $\mathbf{a} \ne \mathbf{c}$, and since $\mathbf{c}$ is in M, $\mathbf{a} \prec \mathbf{c}$ follows (cf. Fig. 2b). Also $\varepsilon \subseteq \delta_2$ follows from $\varepsilon \subseteq \gamma$ and $\gamma \subseteq \delta_2$. Thus, since $\mathbf{a} \in \Delta_{s-q}(\mathbf{b})$, $\mathbf{d} \in \Delta_{s-q}(\mathbf{b})$. But $\mathbf{c} \notin \Delta_{s-q}(\mathbf{b})$ (since $\delta_3 \in [\beta, \gamma]$, $\delta_3 \not\supseteq \alpha$).

Since $\mathbf{a}$ and $\mathbf{d}$ are in the initial segment $A = \Delta_{s-q}(\mathcal{F}_s(\mathbf{b}))$ and $\mathbf{a} \prec \mathbf{c} \prec \mathbf{d}$, it follows ($\mathrm{N}_2$) that $\mathbf{c} \in A$. Since $\mathbf{c} \notin \Delta_{s-q}(\mathbf{b})$, then $\mathbf{c}$ and, therefore, $\mathbf{a}$ are in the initial segment $\Delta_{s-q}(\mathcal{F}_s(\mathbf{b}) \setminus \mathbf{b})$. But this contradicts the definition of $\mathbf{b}$. $\square$

For $0 < k < r_P$ denote

$$W_k = \{(\delta_1, \ldots, \delta_n) \in P_{kn}^n \mid \delta_i \in P_k,\ i = 1, \ldots, n\}.$$

Furthermore, for a poset $(P, \subseteq)$ and $A \subseteq P$ denote by $P[A]$ the poset with the element set A and the induced partial order. Note that if $(P, \subseteq) \in \mathcal{P}$ then for any $\beta \in P$ it holds: $P[I(\beta)] \in \mathcal{P}$. A proof of the next lemma easily follows from the definition of the cartesian product and is omitted.

Lemma 4 *Let $P \in \mathcal{P}$ and $k < r_P$.*

a. Let $\mathbf{a}^1, \mathbf{a}^2 \in W_k$ be distinct elements. Then $I(\mathbf{a}^1) \cap I(\mathbf{a}^2) = \emptyset$ in P^n;

b. $P_i^n = \bigcup_{\mathbf{a} \in W_k} I_i(\mathbf{a})$ for $0 \le i \le k$, where the union is disjoint;

c. For $\mathbf{a} = (\delta_1, \ldots, \delta_n) \in W_k$ the poset $P^n[I(\mathbf{a})]$ is isomorphic to the poset $P[I(\delta_1)] \times \cdots \times P[I(\delta_n)]$.

Let $(P, \subseteq)$ be a ranked poset and $\alpha_1, \alpha_2 \in P_k$, with $k \ge 1$. A path from α_1 to α_2 in the Hasse diagram of P, which consists just of the vertices of P_k and P_{k-1} is called (α_1, α_2)-*path* (if such a path exists). If $\alpha_1 = \alpha_2 = \alpha$, we call α the (α, α)-path as well. We say that the set P_k is *connected* if for any $\alpha_1, \alpha_2 \in P_k$ there exists an (α_1, α_2)-path.

Lemma 5 *Let $(P^{(1)}, \subseteq_1), \ldots, (P^{(n)}, \subseteq_n)$ be some posets from the class $\mathcal{P}$ with $r_{P^{(i)}} = k$, $i = 1, \ldots, n$. Then the k^{th} level of the poset $(\Pi, \subseteq_\times)$ is connected, where $\Pi = P^{(1)} \times \cdots \times P^{(n)}$.*

Proof.
We apply the induction on n. For $n = 1$ the lemma is obviously true, so let $n \geq 2$. Let $\mathbf{a}^1, \mathbf{a}^2 \in \Pi_k$ and $\mathbf{a}^1 = (\alpha_1^1, \ldots, \alpha_n^1)$, $\mathbf{a}^2 = (\alpha_1^2, \ldots, \alpha_n^2)$. We show first that there exists a $(\mathbf{a}^1, \mathbf{b}^1)$-path for some $\mathbf{b}^1 = (\beta_1^1, \ldots, \beta_n^1) \in \Pi_k$ with $r_{P^{(1)}}(\beta_1^1) = 0$.

Indeed, if $r_{P^{(1)}}(\alpha_1^1) = 0$ then we are done. Otherwise, let $\gamma \subset\cdot\ \alpha_1^1$ in $P^{(1)}$. Since $\alpha_i^1 \neq U_{P^{(i)}}$ for all $i \geq 2$, there exists i and $\delta \in P^{(i)}$ such that $\alpha_i^1 \subset\cdot\ \delta$ in $P^{(i)}$. Thus, there exists a $(\mathbf{a}^1, \mathbf{c})$-path, with $\mathbf{c}$ obtained from $\mathbf{a}_1$ by replacing α_1^1 with γ and α_i^1 with δ. Continuing this process until the rank of the first entry is zero, we obtain the vector $\mathbf{b}^1$. Similarly, there exists a $(\mathbf{a}^2, \mathbf{b}^2)$-path for some $\mathbf{b}^2 = (\beta_1^2, \ldots, \beta_n^2)$ with $r_{P^{(1)}}(\beta_1^2) = 0$. Therefore, it is sufficient to show that there exists a $(\mathbf{b}^1, \mathbf{b}^2)$-path.

By the inductive hypothesis there exists a $(\mathbf{b}^1, \mathbf{d})$-path, with $\mathbf{d} = (\beta_1^1, \beta_2^2, \ldots, \beta_n^2)$. Consider the chain

$$\beta_1^1 \subset\cdot\ \gamma_1 \subset\cdot\ \gamma_2 \subset\cdot\ \ldots \subset\cdot\ \gamma_k = U_{P^{(1)}}.$$

Since $\beta_1^1 \subseteq_1 U_{P^{(1)}}$ and $r_{P^{(1)}} = k$, such vertices $\gamma_1, \ldots, \gamma_k$ do exist. Now consider in Π_k the sequence of elements $\mathbf{d}^0 = \mathbf{d}, \mathbf{d}^1, \ldots, \mathbf{d}^k$ with $\mathbf{d}^i$ for $i \geq 1$ obtained from $\mathbf{d}^{i-1} = (\delta_1^{i-1}, \ldots, \delta_n^{i-1})$ by the following: replace δ_1^{i-1} with γ_i, then find maximal index j for which $r_{P^{(j)}}(\delta_j^{i-1}) > 0$ and replace δ_j^{i-1} with ε_j for some $\varepsilon_j \subset\cdot\ \delta_j^{i-1}$ in $P^{(j)}$. Such an element ε_j exists since $r_{P^{(j)}}(\delta_j^{i-1}) > 0$ implies that δ_j^{i-1} is not a leaf (Lemma 3). Then $r_{P^{(1)}}(\delta_1^k) = k$ and, therefore, $r_{P^{(j)}}(\delta_j^k) = 0$ for $j = 2, \ldots, n$. Moreover, there exists a $(\mathbf{d}, \mathbf{d}^k)$-path (in which every other vertex is in the sequence $\mathbf{d}^0, \mathbf{d}^1, \ldots, \mathbf{d}^k$) and, thus, a $(\mathbf{b}^1, \mathbf{d}^k)$-path. Similarly, there exists a $(\mathbf{b}^2, \mathbf{d}^k)$-path and, therefore, a $(\mathbf{b}^1, \mathbf{b}^2)$-path, so the lemma follows. $\square$

This lemma has an immediate corollary, which we need for the proof of the next lemma:

Corollary 1 *For any subset $A \subset \Pi_{k-1}$ there exists an element $\mathbf{a} \in \Pi_{k-1} \setminus A$, such that $\nabla(\mathbf{a}) \cap \nabla(A) \neq \emptyset$.*

Now assuming that for some poset $(P, \subseteq) \in \mathcal{P}$ the poset $(P^n, \subseteq_\times)$ is Macaulay, we establish a structure of the Macaulay order $\preceq$ for bottom levels of P^n. For this fix some k with $0 < k < r_P$ and consider the set W_k. Let $W_k = \{\mathbf{a}^1, \ldots, \mathbf{a}^s\}$, thus, $s = (|P_k|)^n$. Assume that

$$\mathbf{a}^1 \prec \mathbf{a}^2 \prec \cdots \prec \mathbf{a}^s$$

(see Fig. 3). Lemma 4b implies that for any $i \leq k$ the level P_i^n is the disjoint union $\bigcup_{j=1}^s I_i(\mathbf{a}^j)$. In the next lemma we show that the first elements of P_i^n in the order $\preceq$ are the elements of $I_i(\mathbf{a}^1)$. After all the elements of $I_i(\mathbf{a}^1)$ are ordered, the order proceeds with the elements of $I_i(\mathbf{a}^2)$, then with the elements of $I_i(\mathbf{a}^3)$ and so on. An initial segment $\mathcal{F}_i(\mathbf{x})$ of the order $\preceq$ for some $\mathbf{x} \in I_i(\mathbf{a}^3)$ is shown in Fig. 3 by solid lines.

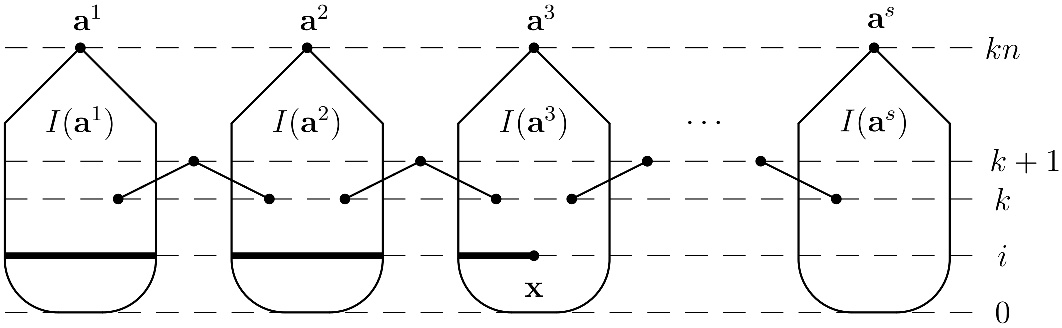

Figure 3: The structure of $k+1$ bottom levels of P^n and of the Macaulay order $\preceq$

Lemma 6 *Suppose that $(P, \subseteq) \in \mathcal{P}$ and that $(P^n, \subseteq_\times)$ is Macaulay for a fixed integer $n \geq 1$. Then for any fixed i and k, $0 \leq i \leq k < r_P$, for any $\mathbf{a}', \mathbf{a}'' \in W_k$, and for any $\mathbf{c} \in I_i(\mathbf{a}')$ and $\mathbf{d} \in I_i(\mathbf{a}'')$ the condition $\mathbf{a}' \prec \mathbf{a}''$ implies $\mathbf{c} \prec \mathbf{d}$.*

Proof.
First let $i = k$. Taking into account Lemma 2, it is sufficient to show that for any element $\mathbf{x} \in P^n_{k-1}$ the conditions $\mathbf{x} \in I(\mathbf{a}^i)$ for some $\mathbf{a}^i \in W_k$ and $I(\mathbf{a}^i) \setminus \mathcal{L}_{k-1}(\mathbf{x}) \neq \emptyset$ imply $\mathbf{y} \in I(\mathbf{a}^i)$, where $\mathbf{y}$ is the predecessor of $\mathbf{x}$ in order $\preceq$ (if such exists).

Assume the contrary, i.e. $\mathbf{x} \in I_{k-1}(\mathbf{a}^i)$ and $\mathbf{y} = \mathrm{pred}(\mathbf{x}) \in I_{k-1}(\mathbf{a}^j)$ for some $j \neq i$. Furthermore, we assume that $\mathbf{x}$ is the greatest element in the order $\preceq$ with this property.

Let $\mathbf{y} = (\xi_1, \ldots, \xi_n)$. Then $r_P(\xi_i) \leq k-1 < k < r_P$, $i = 1, \ldots, n$, imply $\nabla(\mathbf{y}) \in I_k(\mathbf{a}^j)$ and $|\nabla(\mathbf{y})| = n$. Since $\mathcal{L}_{k-1}(\mathbf{x}) \cap I_{k-1}(\mathbf{a}^j) = \emptyset$ by the choice of $\mathbf{x}$, then $\nabla(\mathbf{y}) \cap \nabla(\mathcal{L}_{k-1}(\mathbf{x})) = \emptyset$. Using these assertions one has

$$|\nabla(\mathcal{L}_{k-1}(\mathbf{y}))| = |\nabla(\mathcal{L}_{k-1}(\mathbf{x}))| + |\nabla(\mathbf{y})| = |\nabla(\mathcal{L}_{k-1}(\mathbf{x}))| + n. \tag{1}$$

On the other hand, it follows from Lemmas 5 and 4c that $I_k(\mathbf{a}) = (P^n[I(\mathbf{a})])_k$ is connected. Hence, by Corollary 1, there exists an element $\mathbf{z} \in I_{k-1}(\mathbf{a}^i) \setminus \mathcal{L}_{k-1}(\mathbf{x})$ such that $\nabla(\mathbf{z}) \cap \mathcal{L}_{k-1}(\mathbf{x}) \neq \emptyset$. It then follows from (1) that

$$|\nabla(\mathcal{L}_{k-1}(\mathbf{x}) \cup \mathbf{z})| < |\nabla(\mathcal{L}_{k-1}(\mathbf{x}))| + n = |\nabla(\mathcal{L}_{k-1}(\mathbf{y}))|.$$

This contradicts Lemma 2, since the set $\mathcal{L}_{k-1}(\mathbf{x}) \cup \mathbf{z}$ is not a final segment in the order $\preceq$. Hence, for $i = k$ the lemma is proved. For $i < k$ the lemma follows from Lemma 1 and the property N_2. □

We will often refer to an immediate corollary of this lemma:

Corollary 2 *Let $i \leq k$ and let $\mathbf{a} \in W_k$. Then for the first element $\mathbf{x} \in I(\mathbf{a})$ in the Macaulay order $\preceq$ it holds: $\Delta(\mathcal{F}_i(\mathbf{x}) \setminus \mathbf{x}) \cap \Delta(\mathbf{x}) = \emptyset$.*

Proof of Theorem 1.
It is sufficient to show that if $|\Delta(\alpha)| \geq 2$ for some $\alpha \in P$, then $\alpha = U_P$.

Assume the contrary, i.e that there exists an element $\alpha \in P$ with $r_P(\alpha) < r_P$ and $|\Delta(\alpha)| \geq 2$. Let $k = r_P(\alpha)$ be minimal among all such elements, i.e.

$$|\Delta(x)| \leq 1, \quad \text{whenever} \quad 0 < r_P(x) < k. \tag{2}$$

Let $W_k = \{\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^s\}$. We assume that $\mathbf{a}^1 \prec \cdots \prec \mathbf{a}^s$ and that

$$\mathbf{a}^s = (\alpha_1, \dots, \alpha_n).$$

Our analysis is based on the consideration of set $I(\mathbf{a}^s)$ by taking into account the structure of P^n presented in Lemma 6. Note that for any α_i there exists an element $\beta_i \in P_{k+1}$ with $\alpha_i \subset\cdot\ \beta_i$ and β_i for $i = 1, \dots, n$ is defined uniquely. Furthermore note that for any $(\xi_1, \dots, \xi_n) \in I(\mathbf{a}^s)$ it holds $\xi_i \subseteq \alpha_i$, $i = 1, \dots, n$.

Denote

$$V = \{\mathbf{v} \in P^n_{k+1} \setminus I_{k+1}(\mathbf{a}^s) \mid \Delta(\mathbf{v}) \cap I_k(\mathbf{a}^s) \neq \emptyset\}.$$

Since $\mathbf{v} \notin I(\mathbf{a}^s)$ and $\mathbf{v} \in P^n_{k+1}$, then $\mathbf{v}$ has just one entry of positive rank. Hence, if $\mathbf{v}' \in \Delta(\mathbf{v})$, then $\mathbf{v}' \in I_k(\mathbf{a}^i)$ for some i, $1 \leq i \leq s$. Thus, for any $\mathbf{v} \in V$

$$|\Delta(\mathbf{v}) \cap I_k(\mathbf{a}^s)| = 1. \tag{3}$$

Denote by $\mathbf{a}$ and $\mathbf{b}$ the first vectors of the sets V and $I_{k+1}(\mathbf{a}^s)$ in the order $\preceq$ respectively. Without loss of generality we can assume that

$$\mathbf{a} = (\beta_1, \zeta_2, \dots, \zeta_n), \qquad r_P(\zeta_2) = \cdots = r_P(\zeta_n) = 0.$$

Fact 1: $\mathbf{a} \prec \mathbf{b}$.

Proof. Assume the contrary. Then for any $\mathbf{b}' \in P^n_{k+1}$ preceding $\mathbf{b}$ one has $\mathbf{b}' \notin V$ and, thus, $\Delta(\mathbf{b}') \cap I_k(\mathbf{a}^s) = \emptyset$. Since $\Delta(\mathbf{b}) \subseteq I_k(\mathbf{a}^s)$, then $\Delta(\mathbf{b}') \cup \Delta(\mathbf{b}) = \emptyset$, which implies $\Delta(\mathbf{b}) \cap \Delta(\mathcal{F}_{k+1}(\mathbf{b}) \setminus \mathbf{b}) = \emptyset$. Since $\mathbf{b} \in P^n_{k+1}$ and the rank of each entry of $\mathbf{b}$ is at most k, it follows that $\mathbf{b}$ has at least two entries of positive rank. Thus, $|\Delta(\mathbf{b})| \geq 2$ follows from Lemma 3. Using these assertions and (3) one has

$$\begin{aligned} |\Delta(\mathcal{F}_{k+1}(\mathbf{b}))| - |\Delta(\mathcal{F}_{k+1}(\mathbf{b}) \setminus \mathbf{b})| &= |\Delta(\mathbf{b})| \geq 2 \\ |\Delta(\mathcal{F}_{k+1}(\mathbf{b}) \setminus \mathbf{b}) \cup \mathbf{a})| &= |\Delta(\mathcal{F}_{k+1}(\mathbf{b}) \setminus \mathbf{b})| + 1 < |\Delta(\mathcal{F}_{k+1}(\mathbf{b}))| \end{aligned}$$

This contradicts the property N_1, since $(\mathcal{F}_{k+1}(\mathbf{b}) \setminus \mathbf{b}) \cup \mathbf{a}$ is not an initial segment. □

Denote by $\mathbf{c}$ the first vector of the set $I_k(\mathbf{a}^s)$ in the order $\preceq$. Then $\mathbf{c} \in \Delta(\mathbf{a})$ (cf. Fig 4a). Indeed, if it is not the case, then $\mathbf{c} \prec \mathbf{c}'$ for the first element $\mathbf{c}' \in \Delta(\mathbf{a}) \cap I(\mathbf{a}^s)$. However, this contradicts the fact that the set $\mathcal{F}_{k+1}(\mathbf{a})$ is an initial segment. Therefore, taking into account the form of $\mathbf{a}$, one has

$$\mathbf{c} = (\alpha_1, \zeta_2, \dots, \zeta_n).$$

Fact 2: $|\Delta(\mathbf{c})| \geq 2$.

Proof. First note that $k \geq 2$ and Lemma 3 imply $|\Delta(\mathbf{c})| > 0$. To complete the proof we show that assuming $|\Delta(\mathbf{c})| = 1$ leads to a contradiction. Consider the vector $(\alpha, \ldots, \alpha) \in P_{kn}^n$. Then $(\alpha, \ldots, \alpha) = \mathbf{a}^i$ for some i, $1 \leq i \leq s$. Let $\mathbf{t}$ be the first vector of $I_k(\mathbf{a}^i)$ in the order $\preceq$. Then $|\Delta(\mathbf{t})| \geq 2$. This is obvious if at least two entries of the vector $\mathbf{t}$ are of positive rank. If $\mathbf{t}$ has just one such entry, then $\mathbf{t} \subseteq_\times \mathbf{a}^i$ implies that this entry is α and, thus, $|\Delta(\mathbf{t})| = |\Delta(\alpha)| \geq 2$ by our assumption concerning α.

Now if $i = s$, then $\mathbf{t} = \mathbf{c}$. Thus, $\alpha_1 = \alpha$ and the assertion follows. If $i < s$, then $\mathbf{a}^i \prec \mathbf{a}^s$ and, thus, $\mathbf{t} \prec \mathbf{c}$ by Lemma 6. Consider the set $\mathcal{F}_k(\mathbf{t})$. Applying Corollary 2 with $i = k$ and $\mathbf{x} = \mathbf{t}$ (resp. $\mathbf{x} = \mathbf{c}$), one has

$$\begin{aligned} |\Delta(\mathcal{F}_k(\mathbf{t}))| &= |\Delta(\mathcal{F}_k(\mathbf{t}) \setminus \mathbf{t})| + |\Delta(\mathbf{t})| \geq |\Delta(\mathcal{F}_k(\mathbf{t}) \setminus \mathbf{t})| + 2, \\ |\Delta((\mathcal{F}_k(\mathbf{t}) \setminus \mathbf{t}) \cup \mathbf{c})| &= |\Delta(\mathcal{F}_k(\mathbf{t}) \setminus \mathbf{t})| + |\Delta(\mathbf{c})| = |\Delta(\mathcal{F}_k(\mathbf{t}) \setminus \mathbf{t})| + 1. \end{aligned}$$

Therefore, $|\Delta(\mathcal{F}_k(\mathbf{t}))| > |\Delta((\mathcal{F}_k(\mathbf{t}) \setminus \mathbf{t}) \cup \mathbf{c})|$. However, the set $(\mathcal{F}_k(\mathbf{t}) \setminus \mathbf{t}) \cup \mathbf{c}$ is not an initial segment. This contradicts the property N_1, and completes the proof of the assertion. □

Case 1. Assume $k \geq 2$. Since Fact 2 in combination with Lemma 3 implies $|\Delta_k(\mathbf{c})| \geq 2$, then, applying Corollary 2 with $i = k$ and $\mathbf{x} = \mathbf{c}$, one has

$$|\Delta_k(\mathcal{F}_k(\mathbf{c}))| = |\Delta_k(\mathcal{F}_k(\mathbf{c}) \setminus \mathbf{c})| + |\Delta_k(\mathbf{c})| \geq |\Delta_k(\mathcal{F}_k(\mathbf{c}) \setminus \mathbf{c})| + 2. \tag{4}$$

Now consider the elements $\gamma, \varepsilon \in P$ with $\gamma \subset\cdot\ \alpha_1$ and $\zeta_2 \subset\cdot\ \varepsilon$. Since $r_P(\alpha_1) = k \geq 2$ and $r_P(\zeta_2) = 0$, then $r_P(\gamma) \geq 1$ and $r_P(\varepsilon) = 1$. Denote

$$\mathbf{d} = (\gamma, \varepsilon, \zeta_3, \ldots, \zeta_n) \in I_k(\mathbf{a}^s).$$

Then $\mathbf{c} \prec \mathbf{d}$ follows from the definition of $\mathbf{c}$, and (2) implies $|\Delta_k(\mathbf{d})| = 1$ (cf. Fig 4a). Using (4) and Corollary 2, one has

$$|\Delta_k((\mathcal{F}_k(\mathbf{c}) \setminus \mathbf{c}) \cup \mathbf{d})| = |\Delta_k(\mathcal{F}_k(\mathbf{c}) \setminus \mathbf{c})| + |\Delta(\mathbf{d})| = |\Delta_k(\mathcal{F}_k(\mathbf{c}) \setminus \mathbf{c})| + 1 < |\Delta_k(\mathcal{F}_k(\mathbf{c}))|.$$

This contradicts Lemma 1, because the set $(\mathcal{F}_k(\mathbf{c}) \setminus \mathbf{c}) \cup \mathbf{d}$ is not an initial segment. Thus, if $k > 1$, the theorem is proved.

Case 2. Assume $k = 1$. In this case (2) cannot be used and, thus, we cannot guarantee $|\Delta(\mathbf{d})| = 1$. Now we need a deeper insight into the structure of the poset $(P^n, \subseteq_\times)$. Recall that $\mathbf{a}^s = (\alpha_1, \ldots, \alpha_n)$ and $r_P(\alpha_1) = \cdots = r_P(\alpha_n) = 1$.

Fact 3: $|\Delta(\alpha_i)| \geq 2$, $i = 1, \ldots, n$.

Proof. Consider the set $A = I_1(\mathbf{a}^s)$. Each element of this set has exactly one entry of positive rank, and this entry is α_i for some i, $1 \leq i \leq n$. Let $\mathbf{c} = (\alpha_1, \xi_2, \ldots, \xi_n)$ be the first vector of A in the order $\preceq$ and let $\mathbf{c}' = (\zeta_1, \ldots, \alpha_i, \ldots, \zeta_n)$ be some other element of A. Since $\Delta(\mathcal{F}_1(\mathbf{c}) \setminus \mathbf{c}) \cap \Delta(\mathbf{c}') = \emptyset$ by Corollary 2 (applied with $i = 1$ and $\mathbf{x} = \mathbf{c}'$), then $|\Delta(\mathbf{c})| \leq |\Delta(\mathbf{c}')|$ follows from N_1. This implies $|\Delta(\alpha_1)| = \min_j |\Delta(\alpha_j)|$. Thus,

$$|\Delta(\mathbf{c}')| = |\Delta(\alpha_i)| \geq \min_j |\Delta(\alpha_j)| = |\Delta(\alpha_1)| = |\Delta(\mathbf{c})| \geq 2$$

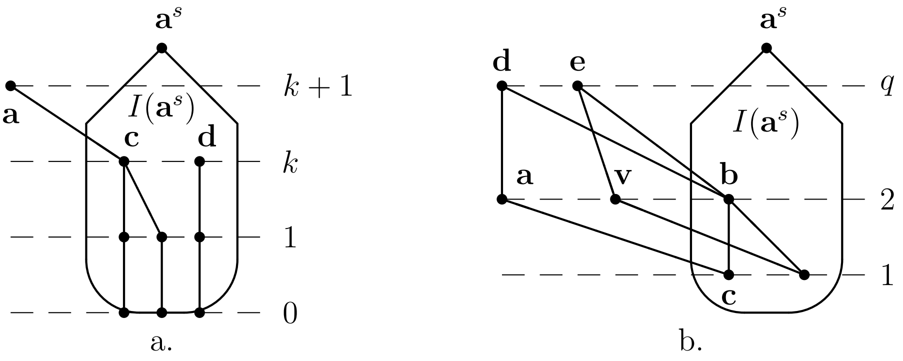

Figure 4: Cases 1 (a.) and 2 (b.) of the proof of Theorem 1

as in Fact 2 and the assertion follows. □

Denote by $\mathbf{b}$ the first element of the set $I_2(\mathbf{a}^s)$ in the order $\preceq$. Without loss of generality we assume that $\mathbf{b}$ is of the form

$$\mathbf{b} = (\alpha_1, \alpha_2, \zeta_3, \ldots, \zeta_n), \qquad r_P(\zeta_3) = \cdots = r_P(\zeta_n) = 0.$$

Let

$$\begin{aligned} \tilde{V} &= \{\mathbf{v} \in P_2^n \setminus I_2(\mathbf{a}^s) \mid \Delta(\mathbf{v}) \cap \Delta(\mathbf{b}) \neq \emptyset\}, \\ \tilde{V}' &= \{\mathbf{v} \in \tilde{V} \mid \mathbf{v} = (\beta_1, \zeta, \zeta_3, \ldots, \zeta_n),\ \zeta \subset\!\cdot\ \alpha_2\}, \\ \tilde{V}'' &= \{\mathbf{v} \in \tilde{V} \mid \mathbf{v} = (\zeta, \beta_2, \zeta_3, \ldots, \zeta_n),\ \zeta \subset\!\cdot\ \alpha_1\}. \end{aligned}$$

Clearly, $\tilde{V} = \tilde{V}' \cup \tilde{V}''$. Denote by $\mathbf{a}$ the first element of the set $\tilde{V}$ in the order $\preceq$. Then $\mathbf{a} \prec \mathbf{b}$ as in Fact 1. Without loss of generality we assume that $\mathbf{a} \in \tilde{V}'$. Taking into account the form of $\mathbf{b}$, for some ζ_2 with $\zeta_2 \subset\!\cdot\ \alpha_2$ one has

$$\mathbf{a} = (\beta_1, \zeta_2, \ldots, \zeta_n).$$

We show that there exists $\mathbf{v} \in \tilde{V}''$ such that $\mathbf{v} \prec \mathbf{b}$ (cf. Fig 4b). Indeed, assume the contrary, i.e. that $\mathbf{v} \succ \mathbf{b}$ for any $\mathbf{v} \in \tilde{V}''$ and consider the set $\mathcal{F}_2(\mathbf{b})$. Then

$$|\Delta(\mathcal{F}_2(\mathbf{b}))| - |\Delta(\mathcal{F}_2(\mathbf{b}) \setminus \mathbf{b})| \geq 2 \tag{5}$$

by Fact 3, since an element $(\zeta, \alpha_2, \zeta_3, \ldots, \zeta_n) \in \Delta(\mathbf{b})$ with $\zeta \subset\!\cdot\ \alpha_1$ cannot be covered by some $\mathbf{b}' \in \tilde{V}'$ and there are at least two such elements. Now, using (3) and (5) we get

$$\begin{aligned} |\Delta((\mathcal{F}_2(\mathbf{b}) \setminus \mathbf{b}) \cup \mathbf{v})| &= |\Delta(\mathcal{F}_2(\mathbf{b}) \setminus \mathbf{b})| + |\Delta(\mathbf{v}) \cap I_1(\mathbf{a}^s)| \\ &\leq |\Delta(\mathcal{F}_2(\mathbf{b}))| - 2 + |\Delta(\mathbf{v}) \cap I_1(\mathbf{a}^s)| \\ &= |\Delta(\mathcal{F}_2(\mathbf{b}))| - 1 < |\Delta(\mathcal{F}_2(\mathbf{b}))|, \end{aligned}$$

which contradicts the property N_1. Therefore, there exists $\mathbf{v} \in \tilde{V}''$ with

$$\mathbf{a} \prec \mathbf{v} \prec \mathbf{b}. \tag{6}$$

Denote $\upsilon = U_P$ and let

$$\mathbf{d} = (\upsilon, \alpha_2, \zeta_3, \dots, \zeta_n), \qquad \mathbf{e} = (\alpha_1, \upsilon, \zeta_3, \dots, \zeta_n).$$

Furthermore let $q = r_P + 1$. It is important to note that any element of P_q^n has at least two entries of positive rank.

Fact 4: ***Let*** $\Delta_{q-2}(\mathbf{w}) \cap I_2(\mathbf{a}^s) \neq \emptyset$ ***for some*** $\mathbf{w} \in P_q^n$***. Then*** $\Delta_{q-1}(\mathcal{F}_q(\mathbf{w})) \cap I_1(\mathbf{a}^s) \supseteq \Delta(\mathbf{b})$***, where the equality holds iff*** $\mathbf{w} \in \{\mathbf{d}, \mathbf{e}\}$***.***

Proof. First consider an element $\mathbf{z} \in \Delta_{q-2}(\mathbf{d})$ different from $\mathbf{a}$ and $\mathbf{b}$. If $\mathbf{z}$ has two entries of positive rank, then the second entry is α_2 and the first entry is not α_1. Thus, $\mathbf{z} \in \mathbf{a}^i$ for some $i < s$ by the definition of $\mathbf{a}^s$. If $\mathbf{z}$ has just one entry of positive rank, then $\mathbf{z} = (\gamma, \eta_2, \zeta_3, \dots, \zeta_n)$ with $\eta_3 \subset\cdot\ \alpha_2$ and $\gamma \in P_2$. Now if $\gamma = \beta_1$, then $\mathbf{z} \in \tilde{V}$ and, $\Delta(\mathbf{z}) \cap I_1(\mathbf{a}^s) \subseteq \Delta(\mathbf{b})$ by (3). If $\gamma \neq \beta_1$, then $\Delta(\mathbf{z}) \cap I_1(\mathbf{a}^s) = \emptyset$. Therefore,

$$\Delta_{q-1}(\mathbf{d}) \cap I_1(\mathbf{a}^s) = \bigcup_{\mathbf{z} \in \Delta_{q-2}(\mathbf{d})} (\Delta(\mathbf{z}) \cap I_1(\mathbf{a}^s)) \subseteq \Delta(\mathbf{b}).$$

On the other hand, $\mathbf{b} \in \Delta_{q-2}(\mathbf{d})$ implies the reverse inclusion. Thus, $\Delta_{q-1}(\mathbf{d}) \cap I_1(\mathbf{a}^s) = \Delta(\mathbf{b})$. Similarly $\Delta_{q-1}(\mathbf{e}) \cap I_1(\mathbf{a}^s) = \Delta(\mathbf{b})$ can be established.

Now assume $\mathbf{w} \notin \{\mathbf{d}, \mathbf{e}\}$ and consider the set $A = \Delta_{q-2}(\mathcal{F}_q(\mathbf{w}))$. We claim that A contains at least one element $\mathbf{b}' \neq \mathbf{b}$ (and, thus, $\mathbf{b}' \succ \mathbf{b}$). Indeed, it is obvious if $\mathbf{b} \notin \Delta_{q-2}(\mathbf{w})$. On the other hand, if $\mathbf{b} \in \Delta_{q-2}(\mathbf{w})$ then $\mathbf{w}$ has at least three entries of positive rank and the claim follows. Since $\mathbf{b}'$ has two entries of positive rank, then $\Delta(\mathbf{b}') \setminus \Delta(\mathbf{b}) \neq \emptyset$. Since A is an initial segment, then $\mathbf{b} \in A$ and we have the assertion. □

Now let $\mathbf{f} \in P_q^n$ be the first element, such that $\Delta_{q-2}(\mathbf{f}) \cap I_2(\mathbf{a}^s) \neq \emptyset$. Then $\mathbf{b} \in \Delta_{q-2}(\mathbf{f})$, since otherwise the set $\Delta_{q-2}(\mathcal{F}_q(\mathbf{f}))$ contains some $\mathbf{b}'$ with $\mathbf{b}' \succ \mathbf{b}$ and, thus, is not an initial segment, which contradicts to N_2. We show that either $\mathbf{f} = \mathbf{d}$ or $\mathbf{f} = \mathbf{e}$ (depending on whether $\mathbf{d} \prec \mathbf{e}$ or $\mathbf{e} \prec \mathbf{d}$ respectively). Assume the contrary, i.e. $\mathbf{f} \notin \{\mathbf{d}, \mathbf{e}\}$. Since $\mathbf{b} \in \Delta_{k-2}(\mathbf{d}) \cap \Delta_{k-2}(\mathbf{c})$, then $\mathbf{f} \prec \mathbf{d}$ and $\mathbf{f} \prec \mathbf{e}$. From Lemma 6 and the definition of $\mathbf{a}^s$ it follows that $I_1(\mathbf{a}^i) \subseteq \Delta_{q-1}(\mathcal{F}_q(\mathbf{f}))$ for all $i < s$. Moreover, $\Delta(\mathbf{b}) \subset \Delta_{q-1}(\mathcal{F}_q(\mathbf{f}))$ by Fact 4. But then

$$\begin{aligned} &\Delta_{q-1}((\mathcal{F}_q(\mathbf{f}) \setminus \mathbf{f}) \cup \mathbf{d}) \subset \Delta_{q-1}(\mathcal{F}_q(\mathbf{f})) \\ &\Delta_{q-1}((\mathcal{F}_q(\mathbf{f}) \setminus \mathbf{f}) \cup \mathbf{e}) \subset \Delta_{q-1}(\mathcal{F}_q(\mathbf{f})), \end{aligned} \tag{7}$$

which contradicts to Lemma 1. Thus, $\mathbf{b} \in \{\mathbf{d}, \mathbf{e}\}$ is established. Note that (7) implies that $\mathbf{d}$ and $\mathbf{e}$ are consecutive elements in the order $\preceq$.

Let $\mathbf{c}$ be the first element of the set $I_1(\mathbf{a}^s)$ in the order $\preceq$.

Fact 5: $\mathbf{c} \in \Delta(\mathbf{b})$.

Proof. Assume the contrary and let $\mathbf{w}$ be the first element of P_q^n in the order $\preceq$ such that $\Delta_{q-1}(\mathbf{w}) \cap I_1(\mathbf{a}^s) \neq \emptyset$. Then $\mathbf{c} \in \Delta_{q-1}(\mathbf{w})$, since otherwise the set $\Delta_{q-1}(\mathcal{F}_q(\mathbf{w}))$ is not an initial segment. Remember that $\mathbf{w}$ has at least two entries of positive rank by the choice of q. Thus, $\Delta_{q-2}(\mathbf{w}) \cap I_2(\mathbf{a}^s) \neq \emptyset$. Therefore, $\mathbf{f} \preceq \mathbf{w}$ follows from the definition of $\mathbf{f}$. On the other hand, since $\Delta_{q-1}(\mathbf{f}) \cap I_1(\mathbf{a}^s) \neq \emptyset$, then $\mathbf{w} \preceq \mathbf{f}$. Hence, $\mathbf{w} = \mathbf{f}$. Now since

$\Delta_{q-2}(\mathcal{F}_q(\mathbf{f})) \cap I_2(\mathbf{a}^s) = \{\mathbf{b}\}$ (cf. the proof of Fact 4), then the set $\Delta_{q-1}(\mathcal{F}_q(\mathbf{f}))$ is not an initial segment, which contradicts Lemma 1. □

It follows from the proof of Fact 5 that the element $\mathbf{f}$ is the first element of P_q^n such that $\Delta_{q-1}(\mathbf{t}) \cap I_1(\mathbf{a}^s) \neq \emptyset$.

Finally, we introduce an element $\mathbf{g} \in P_q^n$ defined as the first element, such that $\mathbf{v} \in \Delta_{q-2}(\mathbf{g})$. Since $\Delta(\mathbf{v}) \cap I_1(\mathbf{a}^s) \neq \emptyset$, then $\Delta_{q-1}(\mathbf{g}) \cap I_1(\mathbf{a}^s) \neq \emptyset$. Taking into account the remark above, we get $\mathbf{f} \preceq \mathbf{g}$. Furthermore, since $\mathbf{f} \in \{\mathbf{d}, \mathbf{e}\}$, since $\mathbf{v} \notin \Delta_{q-2}(\mathbf{d})$ and $\mathbf{v} \in \Delta_{q-2}(\mathbf{e})$, and since $\mathbf{d}, \mathbf{e}$ are consecutive elements in the order $\preceq$, then $\mathbf{g} = \mathbf{e}$.

Now we are ready to obtain a contradiction with the existence of the element α, specified in the beginning of the proof. For that we use (6), which was established assuming the existence of α. First assume that $\mathbf{d} \prec \mathbf{e}$, i.e. $\mathbf{f} = \mathbf{d} \prec \mathbf{e} = \mathbf{g}$ (cf. Fig. 4b). In this case the set $D = \Delta_{q-2}(\mathcal{F}_q(\mathbf{d}))$ is not an initial segment, because $\mathbf{b} \in D$, $\mathbf{v} \prec \mathbf{b}$ by (6) and $\mathbf{v} \notin D$ by the definition of $\mathbf{g}$. This contradicts Lemma 1.

If $\mathbf{e} \prec \mathbf{d}$ and, hence, $\mathbf{f} = \mathbf{g} = \mathbf{e}$, then we have a similar contradiction, as we show that the set $E = \Delta_{q-2}(\mathcal{F}_q(\mathbf{e}))$ is not an initial segment. Since $\mathbf{v} \in E$ and $\mathbf{a} \prec \mathbf{v}$ by (6), it is sufficient to show that $\mathbf{a} \notin E$. Indeed, if we assume $\mathbf{a} \in E$, then the condition $\mathbf{a} \notin \Delta_{q-2}(\mathbf{e})$ implies $\mathbf{a} \in \Delta_{q-2}(\mathbf{h})$ for some $\mathbf{h} \prec \mathbf{e}$. However, $\Delta(\mathbf{a}) \cap I_1(\mathbf{a}^s) \neq \emptyset$ implies $\Delta_{q-2}(\mathbf{h}) \cap I_1(\mathbf{a}^s) \neq \emptyset$, and, thus, $\mathbf{f} \preceq \mathbf{h} \prec \mathbf{e}$ by the definition of $\mathbf{f}$. This contradicts, however, the equality $\mathbf{f} = \mathbf{e}$ and completes the proof of the whole theorem. □

In our forthcoming paper [2] we show that the reverse statement of Theorem 1 is also valid, i.e. that the cartesian product of n posets $Q(k, l)$ is a Macaulay poset for any $n \geq 1$ and any $k \geq 1$, $l \geq 1$.

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