On Oriented Embedding of the Dichotomic Tree into the Hypercube

Sergei L. Bezrukov
Fachbereich Mathematik
Freie Universität Berlin
Arnimallee 2–6, D-14195 Berlin

Abstract

We consider the oriented dichotomic tree and the oriented hypercube. The tree edges are oriented from the root to the leaves, while the orientation of the cube edges is induced by the direction from 0 to 1 in the coordinatewise form. The problem is to embed such a tree with l levels into the oriented n-cube as an oriented subgraph, for minimal possible n. A new approach to such problems is presented, which improves the known upper bound $n/l \leq 3/2$ from [1] to $n/l \leq 4/3 + o(1)$ as $l \to \infty$.

1 Introduction

Denote by B^n the graph of the n-dimensional unit cube. The vertex set of this graph is just the collection of all binary strings of length n, and two vertices are adjacent iff the corresponding sequences differ in one entry only. Let T be a tree. It is easily shown by induction that T is a subgraph of B^n for n sufficiently large. The general question we study here is to find the minimal such n, which we denote by $\mathbf{dim}(T)$ and call the dimension of T.

Such problems arise in the area of computer science dealing with multiprocessor systems [6]. The exact answer depends of course on the structure of the tree T rather than on its simple numerical parameters, e.g., the number of vertices. If one consider trees of bounded vertex degree, which is quite natural for practical applications, one is led to consider the polythomic tree $T^{k,l}$. This is the rooted tree with l levels, where the root has degree k and all the other vertices that are not leaves have degree $k+1$. The dimension of $T^{k,l}$ was studied in [3] (the lower bound) and in [5] (the upper bound), where it is proved that

$$\frac{k \cdot l}{e} \leq \mathbf{dim}(T^{k,l}) \leq \frac{k \cdot l + k + 2l - 2}{2}, \qquad e = 2.718... \tag{1}$$

The lower bound in (1) simply follows from the cardinalities of the sets of vertices at distance at most l from the root, while the upper bound is constructive. Despite several attempts, there have been no improvements of these bounds in the asymptotic sense for arbitrary k, l. For the binary cube it is natural to imagine that the number 2 plays an important role. In accordance with this, let us replace one of the parameters k, l by 2. Then it is known (see [2], [3] respectively) that

$$\begin{aligned} \mathbf{dim}(T^{2,l}) &= l+2 \qquad \text{and} \\ \mathbf{dim}(T^{k,2}) &= \left\lceil \frac{3k+1}{2} \right\rceil . \end{aligned}$$

It is interesting to notice that, although $T^{2,l}$ has $2^{l+1} - 1 < 2^{l+1}$ vertices, the lower bound $\mathbf{dim}(T^{2,l}) \geq l+1$ which follows from the cardinalities is not attainable. Actually in [4] it is proved that one can even find in B^{l+2} two copies of $T^{2,l}$ joined by an edge connecting their roots.

Therefore, in the simplest cases when one of the parameters k, l equals 2, the problem is completely solved. Let us now consider the oriented version of this problem. We orient the edges of $T^{k,l}$ from the root to the leaves, and the edges of B^n as follows. Suppose (v, w) is an edge of B^n such that the sequences v, w differ in the i^{th} entry, where v has 0 and w has 1. Then we orient this edge from v to w. Now we look for an oriented subgraph of B^n isomorphic to $T^{k,l}$. In other words, we consider embeddings of $T^{k,l}$ into B^n such that the i^{th} level of $T^{k,l}$ is embedded into the i^{th} level of B^n, for $i = 0, 1, ..., l$. What is the minimal possible n now? We denote this n by $\vec{\mathbf{dim}}(T^{k,l})$.

It is easy to show that the same lower bound (1), following from the inequality

$$\binom{n}{l} \geq k^l, \tag{2}$$

holds. Indeed there is an even better lower bound [1] for $\vec{\mathbf{dim}}(T^{k,l})$, implied by

$$\binom{n}{l} - \binom{n-k}{l} \geq k^l, \tag{3}$$

but it gives no improvement in the asymptotic sense. As it turns out, the upper bound (1) holds for the oriented case as well, as the construction in [5] provides an oriented embedding.

Let us again consider the case when one of the numbers k, l equals 2. If $l = 2$ then there is no difference between the oriented and non-oriented cases, as without loss of generality one may always assume that the root of $T^{k,2}$ is embedded into the origin of B^n, which forces any embedding to be oriented. It is interesting to note that in this case the lower bound $\vec{\mathbf{dim}}(T^{k,2}) \geq 3k/2$ implied by (3) is asymptotically attained.

The goal of this paper is to study the case $k = 2$. So, we deal with the ordinary dichotomic tree $T^{2,l}$, which we denote T^l for brevity. For this concrete value of k one can get a better lower bound from (2), namely

$$1.2938... \leq \lim_{l\to\infty} \vec{\mathbf{dim}}(T^l)/l. \tag{4}$$

It is easily seen that the trivial upper bound $\vec{\mathbf{dim}}(T^l)/l \leq 2$ equals that given by (1). The best known published upper bound [1] is

$$\lim_{l\to\infty} \vec{\mathbf{dim}}(T^l)/l \leq 3/2. \tag{5}$$

The method of [1] was to find $\vec{\mathbf{dim}}(T^l)$ for $l = 1, ..., 6$, and in particular to prove that T^6 is embeddable into B^9 (here 9/6=3/2). Following this idea one could try to find a clever embedding of T^{l_0} into B^{n_0} for some l_0, n_0, which would imply the upper bound $\lim_{l\to\infty} \vec{\mathbf{dim}}(T^l)/l \le n_0/l_0$. Here we give a table of $n = n(l) = \vec{\mathbf{dim}}(T^l)$ for small values of l.

$l:$	1	2	3	4	5	6	7	8	9	10	11
$n:$	2	4	5	7	8	9	11	12	13	15	16

The entries of this table for $l = 1, ..., 7$ and $l = 10$ are known from [1], while the other three follow from a more detailed analysis, and we mention them here without proof. The values for $l = 9$ and $l = 11$ give us an improvement on (5) as $1.444... = 13/9 < 16/11 < 3/2$. We suspect that it is possible to embed T^{12} into B^{17} (at present we are only able to embed T^{12} into B^{18}), in which case we would be able to improve (5) further to $n/l \le 1.416$ for sufficiently large l. But to find an admissible n as l increases is very difficult, and to get a good upper bound in this way is almost hopeless.

Here we present a new approach for obtaining good bounds for the oriented embedding. Our best result is

Theorem 1 $\lim_{l\to\infty} \vec{\mathbf{dim}}(T^l)/l \le 4/3 = 1.333...$.

If we consider this result in the light of the old techniques from [1], it becomes apparent that, to prove Theorem 1 using the old approach, one would have to prove that T^{3r} can be embedded into B^{4r} for some $r \ge 13$. To demonstrate this, we computed the function $n(l)$ defined by (3) for $l = 1, ..., 39$ and found that the ratio $n(l)/l$ reaches 4/3 for the first time just when $l = 39$.

Let us mention again that, for $l = 1, ..., 11$, $\vec{\mathbf{dim}}(T^l)$ equals the lower bound given by (3). Moreover, $\vec{\mathbf{dim}}(T^{k,l})$ for $k = 1$ or $l = 1$ is also equal to the lower bound implied by the cardinalities. So as yet there are no examples where $\vec{\mathbf{dim}}(T^l)$ is not determined by the bound (3).

Conjecture 1 $\vec{\mathbf{dim}}(T^l)$ *is determined asymptotically by the inequality (2) as* $l \to \infty$.

Conjecture 2 $\vec{\mathbf{dim}}(T^{k,l}) \sim \frac{k \cdot l}{e}$ *as* $k, l \to \infty$.

2 The new approach

Denote by T_i^l $(i = 0, ..., l)$ the i^{th} level of the tree T^l (i.e., the collection of all its vertices at distance i from the root) and by B_i^n $(i = 0, ..., n)$ the i^{th} level of B^n (i.e., the collection of all vertices corresponding to sequences with exactly i ones).

We have the trivial upper bound $\vec{\mathbf{dim}}(T^l)/l \le 2$, and thus we may and shall assume throughout that T_l^l is embedded above the middle level of B^n. Starting from an embedding of T^l into B^n, let us try to embed T_{l+1}^{l+1} using as few additional dimensions as possible. It is clear that we can always succeed using two additional dimensions. The problem is to try to use just one, as we believe in the following conjecture.

Conjecture 3 $\vec{\mathbf{dim}}(T^{l+1}) > \vec{\mathbf{dim}}(T^l)$ *for all* $l \ge 1$.

To use only one additional dimension for T^{l+1} is possible if there exists a matching between the image of T_l^l in B_l^n (which we also denote by T_l^l) and B_{l+1}^n. For example T^2 may be embedded into B^4 with the required matching, which implies $\vec{\mathbf{dim}}(T^3) = 5$. Now $\vec{\mathbf{dim}}(T^4) \geq 7$ simply follows from the cardinalities, and our knowledge about $\vec{\mathbf{dim}}(T^3)$ proves $\vec{\mathbf{dim}}(T^4) = 7$ immediately. When can one guarantee the existence of such a matching?

Let $A \subseteq B_k^n$ and x be a given integer. Define an *x-partition* of A to be a partition of A into s parts A_i with $|A_i| \leq x$ $(i = 1, ..., s)$ such that there is a set $M_x(A) = \{a_i : i = 1, ..., s\}$ of distinct vertices of B_{k+1}^n with a_i adjacent to all vertices of A_i $(i = 1, ..., s)$. Call such a set $M_x(A)$ a *covering set* for the x-partition. In particular if $x = 1$, then a covering set for a 1-partition defines a matching between A and B_{k+1}^n.

If there is an embedding of the tree T^l into B^n in such a way that T_l^l has an x-partition, we write $T^l \rightsquigarrow_x B^n$. The arguments above lead us to the following result.

Proposition 1 *If $T^l \rightsquigarrow_1 B^n$ then $T^{l+1} \rightsquigarrow_2 B^{n+1}$.*

Proof.
Embed T^l into the subcube $x_{n+1} = 0$ in such a way that it has a 1-partition with covering set $A \equiv M_1(T_l^l)$. Set $B = \pi(T_l^l)$ and $C = \pi(A)$, where π is the projection onto the subcube $x_{n+1} = 1$. Now embed T_{l+1}^{l+1} into $A \cup B$ in the obvious way. It is clear that T_{l+1}^{l+1} has a 2-partition with covering set C. ∎

Unfortunately there are examples showing that it is impossible to guarantee that T_l^l has a 1-partition in general, even if $|T_l^l| < |B_{k+1}^n|$. So, the matchings approach does not promise too much, but is the first step towards more general constructions.

Proposition 2
a) If $T^l \rightsquigarrow_2 B^{n_1}$ and $T^k \rightsquigarrow_2 B^{n_2}$ then $T^{l+k} \rightsquigarrow_1 B^{n_1+n_2}$.
b) If $T^l \rightsquigarrow_1 B^{n_1}$ and $T^k \rightsquigarrow_1 B^{n_2}$ then $T^{l+k+1} \rightsquigarrow_2 B^{n_1+n_2}$.

Proof.
a). First we build an embedding of T^l into the subcube B_1 of $B^{n_1+n_2}$ based on the first n_1 coordinates, such that T_l^l has a 2-partition. Now for each vertex $v_i \in T_l^l$ we consider the subcube B_2^i based on the last n_2 coordinates, and embed T^k in each such subcube so that T_k^k has a 2-partition. Thus we get an embedding of T^{l+k} into $B^{n_1+n_2}$. Here we mean that the various embeddings of T^k are isomorphic.

To see that T_{l+k}^{l+k} has a 1-partition, we refer to Fig. 1. In this picture we represent by a, b and c, d vertices of T^{l+k} in the subcubes B_2^i and B_2^j respectively, such that (1) these pairs are in the same parts of the second 2-partition, and (2) the vertex of T^l which is the root of the tree containing a and b is in the same part of the first 2-partition as the vertex corresponding to c and d. Thus from the embedding of T^k into B_2^i and B_2^j, we deduce that there are vertices $e \in B_2^i$ and $f \in B_2^j$ that cover the vertices a, b and c, d respectively. Similarly there exist vertices g and h that cover a, c and b, d respectively. More exactly, the edges $(a, g), (b, h), (c, g), (d, h)$ have directions of edges of the subcube B_1, while the edges $(a, e), (b, e)$ are in B_2^i and $(c, f), (d, f)$ are in B_2^j. The required matching between T_{l+k}^{l+k} and $B_{l+k+1}^{n_1+n_2}$ is depicted by the bold lines.

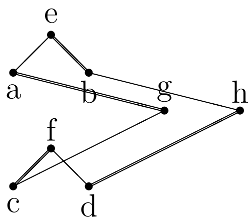


Fig. 1

The proof of b) is quite similar. ∎

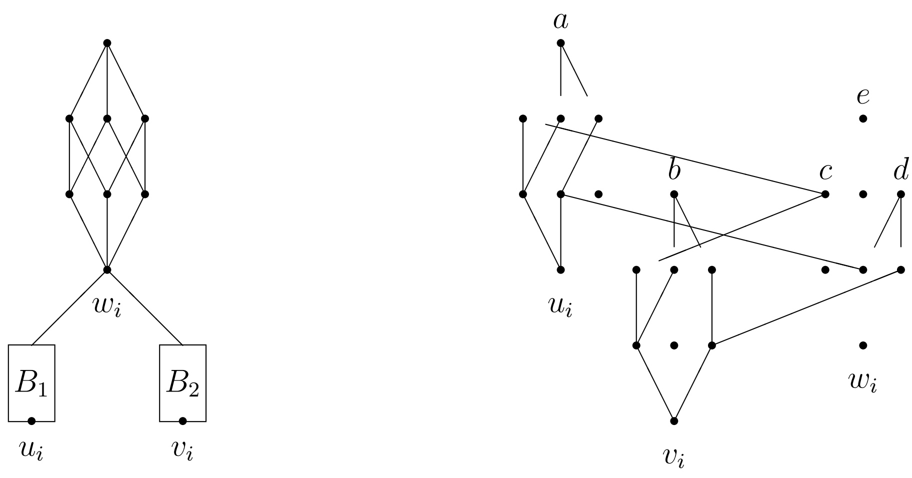
Corollary 1 *If* $T^{l_0} \rightsquigarrow_2 B^{n_0}$ *then* $\vec{\mathbf{dim}}(T^l)/l \le \frac{n_0}{l_0+1/3} + o(1)$ *as* $l \to \infty$.

Proof.
One has $T^{2l_0} \rightsquigarrow_1 B^{2n_0}$ by Proposition 2a. Now we apply Proposition 2b with $k = l = 2l_0$ and get $T^{4l_0+1} \rightsquigarrow_2 B^{4n_0}$. Therefore each time the cube dimension is multiplied by 4, the height of the tree we can embed increases by a multiple of slightly more than 4. More precisely, if the sequences l_i and n_i are defined by $l_i = 4l_{i-1} + 1$ and $n_i = 4n_{i-1}$ $(i = 1, \ldots)$, then we have $T^{l_i} \rightsquigarrow_2 B^{n_i}$ for each i, and $n_i = n_0(l_i + 1/3)/(l_0 + 1/3)$, so $n_i/l_i = \frac{n_0}{l_0+1/3} + o(1)$ as $i \to \infty$. The result follows. ∎

A more detailed analysis of the proof that $\vec{\mathbf{dim}}(T^6) = 9$ shows that $T^6 \rightsquigarrow_2 B^9$, which gives the upper bound $\lim_{l\to\infty} \vec{\mathbf{dim}}(T^l)/l \le 9/(6 + 1/3) \approx 1.421$, but some work is still required. Now we present as the second elementary application of our approach a simple proof of the bound (5).

Proposition 3 *If* $T^l \rightsquigarrow_2 B^n$ *then* $T^{l+2} \rightsquigarrow_2 B^{n+3}$.

Proof.
Embed first the tree T^l into B^n for some n such that T^l_l has a 2-partition with covering set $M_2(T^l_l)$. This is possible by Proposition 1. Now we use this embedding, and its associated 2-partition and covering set, to embed the two extra levels of the dichotomic tree using only three extra dimensions. So, we build the 3-cube growing from each vertex of our n-cube, in particular from each vertex of T^l_l. For each set $\{u_i, v_i\}$ in our 2-partition, let $w_i \in B^n_{l+1}$ be the corresponding vertex in the covering set $M_2(T^l_l)$. This situation is depicted in Fig. 2a, where the rectangles represent the 3-cubes growing from vertices u_i, v_i. The corresponding vertices of these 3-cubes are connected as shown in Fig. 2. We now embed two copies of T^2 rooted in u_i, v_i into this structure, which will provide an embedding of T^{l+2} into B^{n+3}.



a. b.

Fig. 2

Our embedding scheme is shown on Fig. 2b, where we draw the edges of the trees only. Incomplete lines indicate a covering scheme demonstrating that the embedding has a 2-partition.

∎

Now the upper bound (5) follows immediately. We start with an embedding of T^3 into B^5 with a 2-partition, the existence of which was mentioned earlier, and apply Proposition 3. On the i^{th} step of this process we obtain an embedding of T^{3+2i} into B^{5+3i} which implies the upper bound $\lim_{l\to\infty} \vec{\mathbf{dim}}(T^l)/l \leq \lim_{i\to\infty}(3i+5)/(2i+3) = 3/2$.

What is important in our approach is that, given an embedding of T^l into B^n, we construct an embedding of $T^{l+\epsilon}$ into $B^{n+\delta}$, even though $\vec{\mathbf{dim}}(T^\epsilon) > \delta$, which gives the bound $\lim_{l\to\infty} \vec{\mathbf{dim}}(T^l)/l \leq \delta/\epsilon$. We achieve this by using some additional information about the initial embedding, in this case the existence of a 2-partition.

The second important thing is that, in the proof of Proposition 3, it makes no difference which initial embedding we start with. The only thing one needs is a 2-partition, which is in fact easy to guarantee. Indeed, embed T^l into any admissible B^n. Now increase the dimension of the hypercube by 1, adding the subcube $x_{n+1} = 1$. Then each vertex of T^l_l has a neighbor in this subcube, so T^l_l has a 1-partition in B^{n+1}.

Let us finally mention that the proof of the upper bound (5) may be further simplified by using the following result.

Proposition 4 *If* $T^l \rightsquigarrow_1 B^n$, *then* $T^{l+2} \rightsquigarrow_1 B^{n+3}$.

For the proof one has in fact to show that T^2 may be embedded into B^4 so that T^2_2 has a 1-partition, which fact we have already mentioned above. ∎

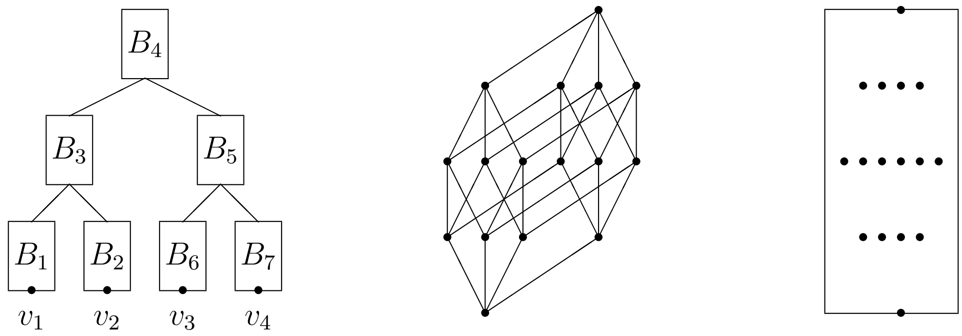
3 Towards better upper bounds

Our general aim is to get a rational upper bound for $\lim_{l\to\infty} \vec{\mathbf{dim}}(T^l)/l$, necessarily exceeding 1.29, using constructions involving low-dimensional cubes only. It seems to be impossible to obtain fully satisfactory results using just x-partitions, with the same x before and after the addition of extra levels. So we need a deeper insight.

For $A \subseteq B_l^n$, $t \geq 2$ and a sequence $(x_1, ..., x_t)$ of positive integers, we say that A can be $(x_1, ..., x_t)$-*partitioned* if there are sets $M_0, M_1, ..., M_t$ such that $M_0 = A$, $M_i \subseteq B_{l+i}^n$ for each i, and, for each i, there is an x_i-partition of M_{i-1} with covering set M_i. If there exists an embedding of T^l into B^n such that T_l^l can be $(x_1, ..., x_t)$-partitioned, we write $T^l \rightsquigarrow_{x_1,...,x_t} B^n$.

Proposition 5 *If* $T^l \rightsquigarrow_{2,2} B^n$ *then* $T^{l+3} \rightsquigarrow_{2,3} B^{n+4}$.

Proof.
Now we have to embed four copies of T^3, rooted in vertices $v_1, ..., v_4$, into the structure depicted in Fig. 3a, where each box represents the 4-cube. The graph of the 4-cube is as shown in Fig. 3b; for convenience we shall normally use the restricted image of it in Fig. 3c. In this image we show just the vertices of B^4, in the same order from left to right as they are shown in Fig. 3b.



a. b. c.

Fig. 3

The embedding we use here is shown in Fig. 4, where, for simplicity, only the subcubes B_1, B_2, B_3 and B_4 are shown, without the edges connecting them. The two copies of T^3 have their roots in vertices v_1, v_2.

Now the top vertices of B_1 and B_2, the vertices of the 3-d level of B_3 and the vertices of the second level of B_4 shown by larger solid circles in Fig. 4 form a covering set M_1 for a 2-partition of this piece of T_{l+3}^{l+3}, and the vertices labeled by asterisks form a covering set for a 3-partition of M_1. This covering scheme is also presented in Fig. 4.

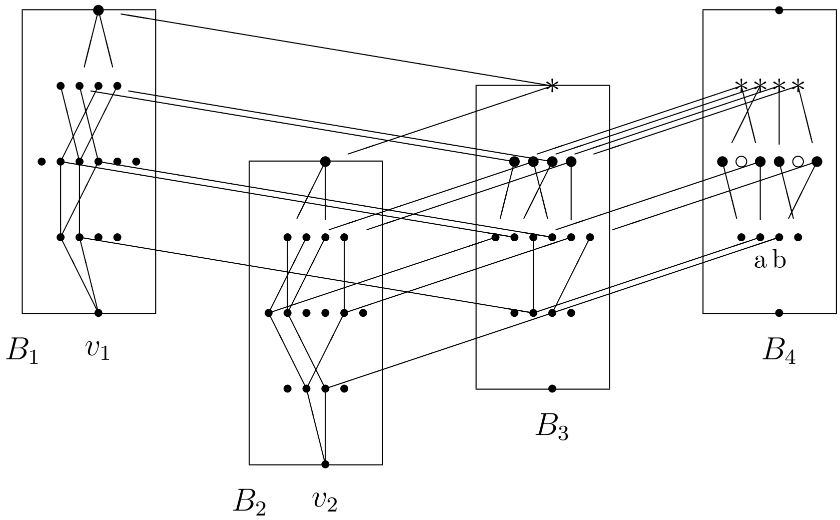


Fig. 4

More about the subcube B_4. In Fig. 4 only two vertices a, b of $T^{l+3} \cap B_4$ are depicted. They correspond to the vertices (0010) and (0100) of B_4 respectively (the commas in vectors are omitted). The vertices of B_4 which cover them correspond to (0110) and (1100) respectively. But we also have to embed four vertices coming from the subcube B_5. In order for all these eight vertices to be distinct, we first embed the two other copies of T^3 into B_5, B_6, B_7 and after that use the isometric transformation of these subcubes defined by the permutation $\binom{1\,2\,3\,4}{3\,4\,1\,2}$ of coordinates. This permutation transforms the four mentioned vertices into (1000), (0001), (0011), and (1001) respectively, which guarantees the correct embedding.

For future reference, note that there are two vertices in the second level of B_4, namely (0101) and (1010), which are not in M_1. They are shown as empty circles in Fig. 4. The Hamming distance between these two vertices is 4. ∎

Using similar techniques one could prove the following properties.

Proposition 6

a) If $T^l \rightsquigarrow_{2,3} B^n$ *then* $T^{l+2} \rightsquigarrow_{2,2} B^{n+3}$*;*
b) If $T^l \rightsquigarrow_{2,2,3} B^n$ *then* $T^{l+3} \rightsquigarrow_{2,2,4} B^{n+4}$*;*
c) If $T^l \rightsquigarrow_{2,2,4} B^n$ *then* $T^{l+3} \rightsquigarrow_{2,3,3} B^{n+4}$*;*
d) If $T^l \rightsquigarrow_{2,3,3} B^n$ *then* $T^{l+2} \rightsquigarrow_{2,2,3} B^{n+3}$*.*

We do not use these properties for the proof of Theorem 1, but think that they are useful for further research. One could combine them with some others in order to get new upper bounds. In particular, Propositions 5, 6a and 6b – 6d respectively imply the following results.

Corollary 2

a) $\lim_{l\to\infty} \vec{\mathbf{dim}}(T^l)/l \le 7/5 = 1.4$*;*
b) $\lim_{l\to\infty} \vec{\mathbf{dim}}(T^l)/l \le 11/8 = 1.375$*.*

Our main result, Theorem 1, is an immediate consequence of the following result.

Proposition 7 *If* $T^l \rightsquigarrow_{2,2,3,4} B^n$ *then* $T^{l+3} \rightsquigarrow_{2,2,3,4} B^{n+4}$.

Proof.
Now we deal with the structure depicted in Fig. 5, where $C_1, ..., C_5$ are 4-cubes and $S_1, ..., S_{12}$ are the structures consisting of seven 4-cubes depicted in Fig. 3a. Each 4-cube C_2, C_3, C_4 is connected with three structures S_i ($i = 4, ..., 12$), as shown in Fig. 5 for the cube C_1. We use the image of B^4 shown in Fig. 3b, and again reduce it to that shown in Fig. 3c.

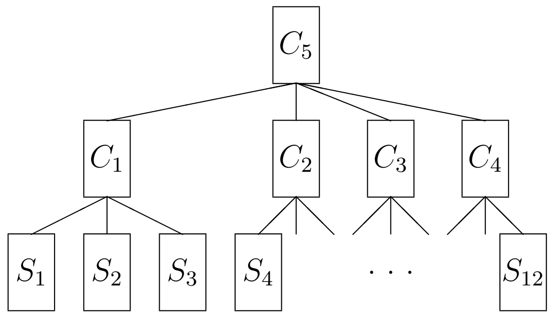


Fig. 5

We start with an embedding of T^l into B^n such that T^l_l can be $(2, 2, 3, 4)$-partitioned, and now embed the three extra levels of our tree by embedding T^3 into each structure S_i as described in the proof of Proposition 5. The role of the remaining 4-cubes in Fig. 5 is to guarantee that T^{l+3}_{l+3} can be $(2, 2, 3, 4)$-partitioned. It was mentioned above that two vertices at distance 4 are free in the subcube B_4 in each structure S_i. Using isometric transformations of the structures S_i we can establish these free vertices to be just (0011) and (1100) in the structure S_i with $i = 0 \pmod 3$, and the vertices (0101),(1010) and (0110),(1001) in the structures S_i with $i = 1 \pmod 3$ and $i = 2 \pmod 3$ respectively. The free vertices are shown as empty circles in the bottom 4-cubes in Fig. 6. These bottom 4-cubes correspond to the subcubes B_4 of the structures S_i (cf. Fig. 3a).

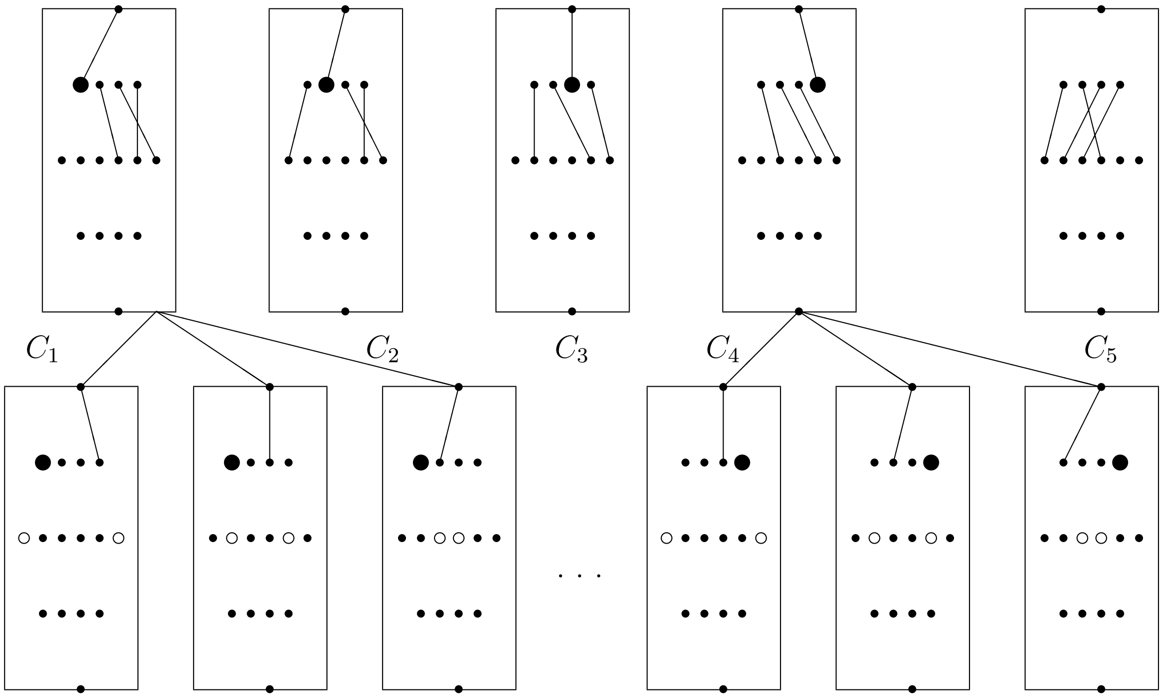
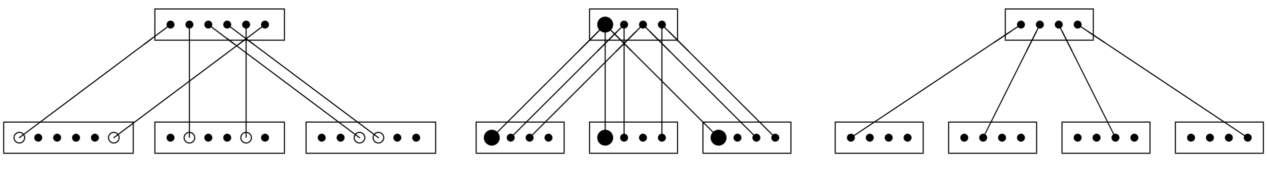


Fig. 6

To prove that our embedding of T_{l+3}^{l+3} can be $(2, 2, 3, 4)$-partitioned, we need to construct sets M_1, M_2, M_3 and M_4 such that M_1 is a covering set for a 2-partition of this section of T_{l+3}^{l+3}, M_2 is a covering set for a 2-partition of M_1, M_3 is a covering set for a 3-partition of M_2, and finally M_4 is a covering set for a 4-partition of M_3.

M_1: we use just the same construction for the set M_1 as in the proof of Proposition 5. This set is shown in Fig. 4 by the large solid circles.

M_2: we take the top vertices of the subcube B_3 to cover the top vertices of the subcubes B_1, B_2 (cf. Fig. 4) in each structure S_i and use all the four vertices in the 3-d level of the subcube B_4 to cover the vertices of the 3-d levels of subcubes B_3, B_5. Now what remains to be done is to cover the four solid vertices in the second level of the subcube B_4 in each structure S_i (see Fig. 4) by the six vertices in the second level of the subcubes C_i $(i = 1, ..., 4)$ (cf. Fig. 5). The covering scheme is explained in Fig. 7a. In this figure we represent the six vertices of C_1 by the top block and the second levels of B_4 in S_1, S_2, S_3 by the three bottom blocks (for other C_i and S_i, the principle is the same). Each vertex of the top block is incident (in B^n) to the three corresponding vertices of the bottom blocks, but we have to choose only two edges to cover all the solid vertices. Remove now the edges shown in Fig. 7a. Then the remaining edges between the top and bottom blocks form the required covering.



a. b. c.

Fig. 7

M_3: as constructed above, M_2 consists of the top vertices of the subcubes B_3, the second levels of the subcubes B_4 (cf. Fig. 4) and the second levels of the subcubes $C_1, ..., C_4$. Now we have to cover all these vertices by the top vertices of the subcubes B_4, the third levels of $C_1, ..., C_4$ and the second level of C_5, in such a way that no vertex is matched to more than three from M_2.

Consider now the subcube C_1 and the subcubes B_4 in the structures S_1, S_2, S_3. We cover the top vertices of the subcubes B_3, B_5 from the top vertex of the subcubes B_4 in each S_i (cf. Fig. 4) and use the third edge to cover one of the three vertices in the 3-d levels of B_4 as shown in Fig. 6 (these three vertices are depicted by small circles). In order to cover the remaining three vertices of B_4's (depicted by large circles in Fig 6) we use the 3-d level of C_1 and the covering scheme as shown in Fig. 7b. In this picture the leftmost (large) vertex has degree three while all the other vertices have degree two. We use the remaining three edges incident to these vertices to cover some three vertices in the second level of C_1, as shown in Fig. 6.

Therefore the vertex of each subcube B_4 represented by the largest circle (see Fig. 6) in the structures S_i $(i = 1, 2, 3)$ plays a particular role. In other structures we use a similar principle, and the corresponding vertices of the B_4's are represented in Fig. 6 by large circles. Of course, one then has to correct the covering scheme in the subcubes C_2, C_3, C_4, which we do in accordance with Fig. 6.

Considering now the 4-cubes $C_1, ..., C_4$ in Fig. 6, notice that the two rightmost vertices in their second levels are already covered from the 3-d levels and just one of the other four vertices is also covered. In order to cover the remaining three vertices in each subcube $C_1, ..., C_4$ we use the leftmost four vertices of C_5 and the covering graph depicted in Fig. 7c. Each vertex of the top block is incident to the corresponding vertex in each bottom block, and removing the depicted edges we get the required covering.

M_4: finally we construct M_4 from the top vertices of the subcubes $C_1, ..., C_4$ and the third level of C_5. Indeed, cover the top vertices of the subcubes B_4 in each structure S_j from the top vertex of the corresponding subcube C_i $(i = 1, ..., 4)$. Thus we used three edges for each C_i. The 4^{th} edge is used to cover the large vertices of the 3-d level in each C_i, as shown at the top of Fig. 6. The remaining (small) vertices of C_i $(i = 1, ..., 4)$ we cover from the 3-d level of C_5 using three edges, with a covering scheme similar to that in Fig. 7c. Now each vertex of the third level of C_5 is used to cover some three vertices of M_3, and we use the 4^{th} edge incident to each of them to cover the leftmost four vertices in the second level of C_5 (see Fig. 6). $\blacksquare$

We hope that, using similar techniques, one can operate with larger graphs, and construct an embedding of 10 extra levels in 13 extra dimensions and finally prove the following.

Conjecture 4 $\lim_{l\to\infty} \vec{\mathbf{dim}}(T^l)/l \leq 13/10 = 1.30$.

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