# Properties of Graded Posets Preserved by Some Operations


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**Abstract**

We answer the following question: Let $P$ and $Q$ be graded posets having some property and let $\circ$ be some poset operation. Is it true that $P \circ Q$ has also this property? The considered properties are: being Sperner, a symmetric chain order, Peck, LYM, and rank compressed. The studied operations are: direct product, direct sum, ordinal sum, ordinal product, rankwise direct product, and exponentiation.


## 1 Introduction and overview

Throughout we will consider finite *graded* partially ordered sets, i.e. finite posets in which every maximal chain has the same length. For such posets $P$ there exists a unique function $r : P \mapsto I\!N$ (called *rank function*) and a number $m$ (called *rank of* $P$), such that $r(x) = 0$ ($r(x) = m$) if $x$ is a minimal (resp., maximal) element of $P$, and $r(y) = r(x) + 1$ if $y$ covers $x$ in $P$ (denoted $x \lessdot y$). The set $P_{(i)} := \{x \in P : r(x) = i\}$ is called $i$*-th level* and its cardinality $|P_{(i)}|$ the $i$*-th Whitney number*. If $S$ is a subset of $P$, let $r(S) := \sum_{x \in S} r(x)$, in particular $r(P) := \sum_{x \in P} r(x)$. Let us emphasize, that $r(P)$ is here not the rank of $P$.

A symmetric chain is a chain of the form $C = (x_0 \lessdot x_1 \lessdot \ldots \lessdot x_s)$, where $r(x_0) + r(x_s) = m$. A subset $A$ of $P$ is called a $k$*-family*, if there are no $k+1$ elements of $A$ lying on one chain in $P$. Further, $F \subseteq P$ is called a *filter*, if $y \geq x \in F$ implies $y \in F$, and $I \subseteq P$ is said to be an *ideal*, if $y \leq x \in I$ implies $y \in I$. Let $d_k(P) := \max\{|A| : A \text{ is a } k\text{-family}\}$ and $w_k(P)$ denotes the largest sum of $k$ Whitney numbers. Obviously, $w_k(P) \leq d_k(P)$, for $k \geq 1$. The poset $P$ is said to be:

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i) *Sperner* (S), if $d_1(P) = w_1(P)$,

ii) *symmetric chain order* (SC), if $P$ has a partition into symmetric chains,

iii) *Peck*, if $d_k(P) = w_k(P)$, for $k \geq 1$, and $|P_{(0)}| = |P_{(m)}| \leq |P_{(1)}| = |P_{(m-1)}| \leq \cdots \leq |P_{(\lfloor m/2 \rfloor)}| = |P_{(\lceil m/2 \rceil)}|$,

iv) *LYM*, if $\sum_{x \in A} \frac{1}{|P_{(r(x))}|} \leq 1$ for every antichain $A$ of $P$,

v) *rank compressed* (RC), if $\mu_r := \frac{r(F)}{|F|} \geq \frac{r(P)}{|P|} =: \mu_P$ for every filter $F \neq \emptyset$ of $P$.

Since $F$ is a filter iff $P \setminus F$ is an ideal, one can define equivalently

v)' $P$ is rank compressed, if $\mu_I := \frac{r(I)}{|I|} \leq \mu_P$ for every ideal $I \neq \emptyset$ of $P$.

Let us mention that P.Erdös [6] proved already in 1945 that finite Boolean lattices are Peck.

In the following we will study which of these properties are preserved by usual poset operations, i.e. the question is: if $P$ and $Q$ have some property, is it true that $P \circ Q$ has this property either (here $\circ$ is some operation)?

Throughout let $m$ (resp., $n$) be the rank of $P$ (resp., $Q$). If it is not clear from the context whether $r$ is the rank function of $P$, $Q$, or $P \circ Q$ we will write $r_P$, $r_Q$, $r_{P \circ Q}$, respectively.

A widely studied operation is the *direct product* $P \times Q$, i.e. the poset on the set $\{(x, y) : x \in P \text{ and } y \in Q\}$, such that $(x, y) \leq (x', y')$ in $P \times Q$ if $x \leq_P x'$ and $y \leq_Q y'$. It is well-known, that the direct product preserves the properties SC (de Bruijn et al. [2] and Katona [10]), Peck (Canfield [3]), and RC (Engel [4]), and it does not preserve the properties S and LYM (but with an additional condition it does (Harper [8] and Hsieh, Kleitman [9])), see Figure 1.

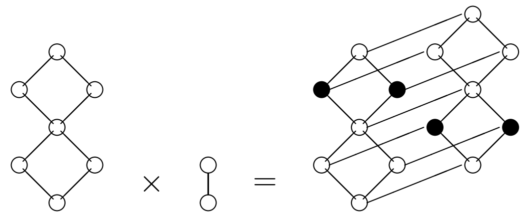


Fig. 1

A simple operation is the *direct sum* $P + Q$, i.e. the poset on the union $P \cup Q$, such that $x \leq y$ in $P + Q$ if either $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$. In order to obtain again a graded poset we will suppose here $m = n$. Then it is easy to see, that SC and Peck properties are preserved, but S, LYM, and RC not, see Figure 2.

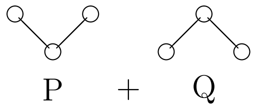


Fig. 2

Another easy operation is the *ordinal sum* $P \oplus Q$, i.e. the poset on the union $P \cup Q$, such that $x \leq y$ in $P \oplus Q$, if $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$, or $x \in P$ and $y \in Q$. To draw the Hasse diagram of $P \oplus Q$, put $Q$ above $P$ and connect each maximal element of $P$ with each minimal element of $Q$. Then it is obvious, that properties S and LYM are preserved (note that any antichain in $P \oplus Q$ is either contained completely in $P$ or completely in $Q$), and also property RC is preserved (see Theorem 1), but properties SC and Peck are not preserved, see Figure 3.

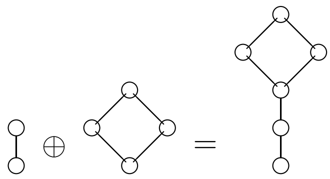


Fig. 3

An interesting operation is the *ordinal product* $P \otimes Q$, i.e. the poset on the set $\{(x, y) : x \in P \text{ and } y \in Q\}$, such that $(x, y) \leq (x', y')$ in $P \otimes Q$, if $x = x'$ and $y \leq_Q y'$, or $x <_P x'$. To draw the Hasse diagram of $P \otimes Q$, replace each element $x$ of $P$ by a copy $Q_x$ of $Q$, and then connect every maximal element of $Q_x$ with every minimal element of $Q_y$ whenever $y$ covers $x$ in $P$. In Theorem 2 we will prove, that properties S, LYM, and RC are preserved. Figure 4 shows that properties SC and Peck are not preserved.

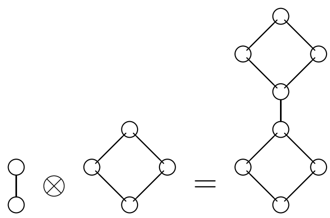
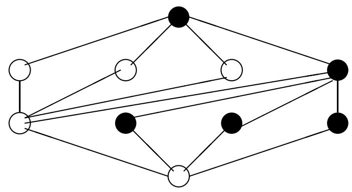


Fig. 4

Studying posets like square submatrices of a square matrix, Sali [11] introduced the *rankwise direct product* $P \times_r Q$. We will suppose here again $m = n$. Then $P \times_r Q$ is the subposet of $P \times Q$, induced by $\bigcup_{i=0}^{m} P_{(i)} \times Q_{(i)}$. Sali [11] showed, that properties SC, Peck, and LYM are preserved and gave an example that property S is not preserved. Here we present an example, which shows that also RC property is not preserved. Look at the poset $P$ of Figure 5, which is easily seen to be rank compressed.



P

Fig. 5

The indicated elements form a filter $F$. Now it is easy to see that the filter $F \times_r F$ in $P \times_r P$ does not verify the filter inequality of v).

Finally, we will consider also the *exponentiation* $Q^P$, i.e. the poset on the set of all order-preserving maps $f : P \mapsto Q$ (that is, $x \leq_P y$ implies $f(x) \leq_Q f(y)$), such that $f \leq g$ if $f(x) \leq_Q g(x)$ for all $x \in P$. In Theorem 3 we will prove that none of the 5 properties is preserved.

# 2 Main results

**Theorem 1** *If $P$ and $Q$ are rank compressed, then $P \oplus Q$ is rank compressed either.*

**Proof.** Obviously, if $y$ has rank $i$ in $Q$, then it has rank $i + m + 1$ in $P \oplus Q$. Hence $\mu_{P\oplus Q} = \frac{r_P(P)+r_Q(Q)+(m+1)|Q|}{|P|+|Q|}$. Let $I$ be an ideal in $P \oplus Q$ and $I \neq \emptyset$.

*Case* 1. Assume $I \cap Q = \emptyset$. Since $\mu_P \leq \mu_{P\oplus Q}$, and $\mu_I \leq \mu_P$ as $p$ is rank compressed, it follows $\mu_I \leq \mu_{P\oplus Q}$.

*Case* 2. Let no $I \cap Q \neq \emptyset$. Then $P \subseteq I$ and $\tilde{I} := Q \cap I$ is an ideal in $Q$. One has $|I| = |P| + |\tilde{I}|$, $r(I) = r_P(P) + r_Q(\tilde{I}) + (m+1)|\tilde{I}|$. Therefore, $\mu_I \leq \mu_{P\oplus Q}$ is equivalent to

$$\Big(|\tilde{I}|r_Q(Q) - |Q|r_Q(\tilde{I})\Big) + |Q \setminus \tilde{I}|\Big((m+1)|P| - r(P)\Big) + |P|r(Q \setminus \tilde{I}) \geq 0.$$

This inequality is true, since $Q$ is rank compressed and any element of $P$ has rank at most $m$.

∎

**Theorem 2** *If $P$ and $Q$ are Sperner or LYM or rank compressed, then $P \otimes Q$ is resp., Sperner or LYM or rank compressed either.*

**Proof.** Let $A$ be an antichain in $P \otimes Q$. Denote $A_x = \{y \in Q : (x,y) \in A\}$ and $\tilde{A} = \{x \in P : A_x \neq \emptyset\}$. Obviously, $\tilde{A}$ and $A_x$ are antichains in $P$ and $Q_x$, respectively.

If $P$ and $Q$ are Sperner, then

$$|A| = \sum_{x\in\tilde{A}} |A_x| \leq \sum_{x\in\tilde{A}} w_1(Q) = |\tilde{A}|w_1(Q) \leq w_1(P)w_1(Q) = w_1(P \otimes Q),$$

hence $P \otimes Q$ is Sperner.

Now let $P$ and $Q$ be LYM. Obviously, the level containing $(x, y)$ has $|P_{(r(x))}||Q_{(r(x))}|$ elements. We have

$$\sum_{(x,y)\in A} \frac{1}{|(P\otimes Q)_{(r(x,y))}|} = \sum_{(x,y)\in A} \frac{1}{|P_{(r(x))}||Q_{(r(y))}|} = \sum_{x\in\tilde{A}} \frac{1}{|P_{(r(x))}|} \sum_{y\in A_x} \frac{1}{|Q_{(r(y))}|} \le 1.$$

Finally, let $P$ and $Q$ be rank compressed. Let $I$ be an ideal in $P \otimes Q$ and $A$ be the set of maximal elements of $I$ (note that $A$ is an antichain). We use the notation $\tilde{A}$ from above and define further $I_x := I \cap Q_x$, $F_x := Q_x \setminus I_x$, $\tilde{I} := \{x \in P : I_x \neq \emptyset\}$. Then $I_x$ ($F_x$) is an ideal (resp., a filter) in $Q_x$ and $\tilde{I}$ is an ideal in $P$. It is easy to see that:

$$\begin{aligned}
|I| &= |\tilde{I}||Q| - \sum_{x\in\tilde{A}} |F_x|,\\
r(I) &= |\tilde{I}|r_Q(Q) + (n+1)|Q|r_P(\tilde{I}) - \sum_{x\in\tilde{A}} \Big(r_Q(F_x) + (n+1)|F_x|r_P(x)\Big)\\
|P\otimes Q| &= |P||Q|\\
r(P\otimes Q) &= |P|r_Q(Q) + (n+1)|Q|r_P(P).
\end{aligned}$$

Now $\frac{r(I)}{|I|} \le \frac{r(P\otimes Q)}{|P\otimes Q|}$ iff

$$|P| \sum_{x\in\tilde{A}} \Big(|F_x|r_Q(Q) - |Q|r_Q(F_x)\Big) \le (n+1)|Q|\times$$

$$\left[|Q|\Big(|\tilde{I}|r_P(P) - |P|r_P(\tilde{I})\Big) + \sum_{x\in\tilde{A}} \Big(|P|r_P(x) - r_P(P)\Big)|F_x|\right].$$

The LHS is not greater than 0 since $Q$ is rank compressed. So it is sufficient to verify, that the formula in brackets is not smaller than 0. Denote $\tilde{A}' = \{x \in \tilde{A} : |P|r_P(x) - r_P(P) \le 0\}$. Since one can omit the positive summands in the formula and in view of $|F_x| \le |Q_x|$ it is enough to show that

$$|\tilde{I}|r_P(P) - |P|r_P(\tilde{I}) + \Big(|P|r_P(\tilde{A}') - |\tilde{A}'|r_P(P)\Big) \ge 0,$$

which is equivalent to

$$|\tilde{I} \setminus \tilde{A}'|r_P(P) - |P|r_P(\tilde{I} \setminus \tilde{A}') \ge 0.$$

This inequality is true, since $P$ is rank compressed and $\tilde{I} \setminus \tilde{A}'$ is an ideal in $P$. ∎

Let $P^l$ be the direct product of $l$ copies of $P$. The investigation of rank compressed posets was initiated by the following result of Alekseev [1]:

$$P \text{ is rank compressed iff } d_1(P^l) \sim w_1(P^l) \text{ as } l \mapsto \infty. \tag{1}$$

Moreover, from the Local Limit Theorem of Gnedenko one can easily derive

$$w_1(P^l) \sim \frac{|P|^l}{\sqrt{2\pi l}\sigma_P} \quad \text{if } P \text{ is not an antichain,}$$

where $\sigma_P^2 = \frac{1}{|P|} \sum_{x \in P} r^2(x) - \mu_p^2$ (see Engel, Gronau [5]).

**Remark.** Straight-forward computations give us the following results:

$$\begin{aligned}
\sigma^2_{P\times Q} &= \sigma^2_P + \sigma^2_Q, \\
\sigma^2_{P\otimes Q} &= \sigma^2_Q + (n+1)^2\sigma^2_P, \\
\sigma^2_{P\oplus Q} &= \frac{|P||Q|}{(|P|+|Q|)^2}\Big(\mu_Q + (m+1-\mu_P)\Big)^2 + \frac{1}{|P|+|Q|}\Big(|P|\sigma^2_p + |Q|\sigma^2_Q\Big).
\end{aligned}$$

∎

**Theorem 3** *The exponentiation does not preserve any of the properties S, SC, Peck, LYM, or RC.*

**Proof.** First take $P$ and $Q$ from Figure 6.

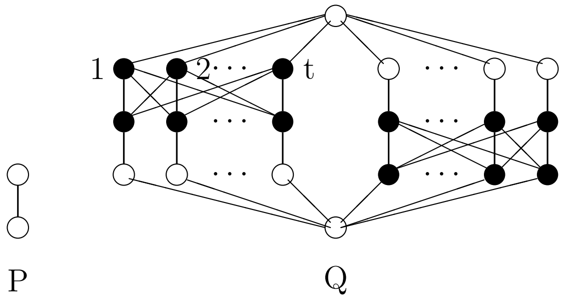



Fig. 6

Here we mean that the complete bipartite graphs $K_{t,t}$ are forms on the indicated vertices.

Obviously, $P$ and $Q$ have properties S, SC, Peck, LYM, and RC. Now $Q^P$ is isomorphic to the subposet of $Q \times Q$ induced by the set $\{(x, y) : x \leq_Q y\}$. Consider the ideal

$$I := \{(x, y) \in Q^P : x \leq i \text{ and } y \leq i \text{ for some } i = 1, \ldots, t\}.$$

Easy calculations give us

$$\mu_I = \frac{9t^2 + 21t}{2t^2 + 7t + 1} \quad \text{and} \quad \mu_{Q^P} = 4.$$

But $\mu_I > \mu_{Q^P}$ iff $t \geq 8$. Hence, $Q^P$ is not rank compressed if $t \geq 8$, and consequently has not properties SC, Peck, LYM, since these properties imply property RC (see [5]).

Finally let $t \geq 8$ and denote $P' = P + \cdots + P$ ($l$ times). Again, $P'$ has all of the properties above. It is known (see Stanley [12]), that $Q^{P+\cdots+P} \cong Q^P \times \ldots \times Q^P$, hence $Q^{P'} \cong (Q^P)^l$. Since $Q^P$ is not rank compressed, by (1)

$$d_1\Big(Q^{P'}\Big) = d_1\Big((Q^P)^l\Big) > w_1\Big((Q^P)^l\Big) = w_1\Big(Q^{P'}\Big)$$

if $l$ is sufficiently large. Thus, $Q^{P'}$ is not Sperner. ∎

Concerning the exponentiation, let us mention that if $Q$ is a distributive lattice, then so is $Q^P$. Since distributivity implies rank compression (see [5]), in a lot of cases the exponentiation provides a rank compressed poset. In particular, if $Q$ is a two-element chain, $Q^P$ is isomorphic to the lattice of ideals of $P$, which is consequently rank compressed for any poset $P$.

# 3 Summary

In the following table we have summarized which of the considered properties are preserved and which not:

| | $P+Q$ $m=n$ | $P \oplus Q$ | $P \times Q$ | $P \otimes Q$ | $P \times_r Q$ $m=n$ | $Q^P$ |
|---|---|---|---|---|---|---|
| Sperner | no | yes | no | yes | no | no |
| Symm. chain | yes | no | yes | no | yes | no |
| Peck | yes | no | yes | no | yes | no |
| LYM | no | yes | no | yes | yes | no |
| Rank compr. | no | yes | yes | yes | no | no |

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