On Superspherical Graphs

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Abstract

In this paper we consider two conjectures of H.M. Mulder [M] and give partial solutions for them. First we prove that a triangle free superspherical graph is interval regular provided it satisfies an additional (not too strong) condition. Furthermore, we show that a spherical interval regular graph is interval monotone.

1 Introduction

Graphs that are well-structured or highly symmetrical are studied extensively. In particular hypercubes and graphs that are close to hypercubes draw much attention. In his book [M] H.M. Mulder put forward the following conjecture (see the definitions below).

Conjecture 1.1 *An interval regular graph is interval monotone.*

Furthermore, he proposed the following.

Conjecture 1.2 *A triangle free superspherical graph is interval regular.*

In the present paper we prove two related theorems. In order to formulate the results we need some definitions.

Definition 1.3 *Let $G = (V, E)$ be a finite graph and let $u, v \in V$. The* interval *$I(u, v)$ is the subgraph of G induced by the set of all vertices lying on a shortest path between u and v. The* length *of the interval is the distance $d(u, v)$ of u and v.*

Let us denote the d-dimensional hypercube by $\mathcal{B}_d$. So $\mathcal{B}_d$ is the graph whose vertex set is $\{0, 1\}^d$ and two vertices are connected by an edge iff they differ in exactly one position.

Definition 1.4 *Let $G = (V, E)$ be a finite connected graph. G is called* interval regular *if for any $u, v \in V$ the subgraph induced by the set of edges between levels in the interval $I(u, v)$ is a hypercube $\mathcal{B}_d$ with $d = d(u, v)$.*

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Note that this definition is different from that of [M], but the two are equivalent. Furthermore, $I(u,v)$ need not be isomorphic to $\mathcal{B}_d$ even in the case of an interval regular G. Let us denote the property that *the subgraph induced by the set of edges between levels in the interval* $I(u,v)$ *is a hypercube* $\mathcal{B}_d$ *with* $d = d(u,v)$ by $I(u,v) \bowtie \mathcal{B}_d$. The next definition deals with convexity properties of intervals.

Definition 1.5 *Let* $G = (V,E)$ *be a finite graph and let* $A \subset V$. *We say that* A *is* convex *if for every* $x, y \in A$ *the vertices of* $I(x,y)$ *are contained in* A. G *is* interval monotone *if each interval of* G *is convex.* G *is said to have the* quadrangle property, *if for every interval* $I(u,v)$ *of* G *and every* $x, y \in I(u,v)$ *such that* $d(x,u) = 1$ *and* $d(y,u) = 1$, *there exists a* $u \neq z \in I(u,v)$ *such that* $d(z,x) = d(z,y) = 1$.

One more definition is needed.

Definition 1.6 *Let* $G = (V,E)$ *be a finite graph and let* $u, v \in V$ *such that* $d(u,v) = d$. *Let* $x, y \in I(u,v)$. *We say that* x *and* y *are* diametrical *if* $d(x,y) = d(u,v) = d$. G *is called* spherical *if for every point in each of its intervals there exists at least one diametrical point in that particular interval.* G *is* superspherical *if spherical and for every point the diametrical pair is unique in every interval.*

If x and y are diametrical in some interval, then y is often called the *opposite* or *complement* of x in that interval. It is easy to see that $\mathcal{B}_d$ is an example of superspherical graphs. However, a superspherical graph need not be a hypercube as the following example shows. Let $\mathcal{B}_d(1,d)$ denote the graph whose vertex set is $\{0,1\}^d$ and two vertices are connected by an edge iff they differ in exactly one or d positions. Then $\mathcal{B}_{2t}(1,2t)$ is a superspherical graph.

Now we can formulate the main results.

Theorem 1.7 *If a triangle free superspherical graph* G *satisfies the quadrangle property, then it is interval regular.*

Theorem 1.8 *If an interval regular graph* G *is spherical, then it is interval monotone.*

As an immediate corollary we obtain the following.

Corollary 1.9 *For triangle free superspherical graphs the following three conditions are equivalent:*

(i) G *satisfies the quadrangle property*
(ii) G *is interval monotone*
(iii) G *is interval regular.*

∎

We mention that Havel and Liebl proved that a bipartite superspherical graph is in fact isomorphic to a hypercube [HL].

2 Proofs

The proof of Theorem 1.7 consists of several small steps. We use induction on d. Let α be a neighbor of v in $I(u,v)$. Let $\bar{\alpha}$ be its unique diametrical pair. Let level i consist of vertices in $I(u,v)$ of distance i from u. For $z \in I(u,v)$ let $l(z)$ denote the level of z. The first step is the case $d = 2$.

Step 1 *If x and y are such that $d(x,y) = 2$, then $I(x,y) \cong \mathcal{B}_2$.*

Proof.
$I(x,y)$ consists of common neighbors of x and y besides x and y because $d(x,y) = 2$. By the triangle free property there does not exist edge between two different neighbors of x. There is at least one vertex $z \in I(x,y)$, so there must exist its diametrical pair, say w. Now, if there was any other vertex in $I(x,y)$, then z would have more than one diametrical pair, a contradiction. ∎

Step 2 $I(u,\alpha) \cap I(\bar{\alpha},v) = \emptyset$.

Proof.
Suppose in contrary that there exists β of level i in $I(u,v)$ such that $\beta \in I(u,\alpha) \cap I(\bar{\alpha},v)$. Then we have $d(\bar{\alpha},\beta) = i-1$ and $d(\beta,\alpha) = d-i$ so

$$d(\alpha,\bar{\alpha}) \leq d(\bar{\alpha},\beta) + d(\beta,\alpha) = i-1+d-i < d$$

would hold, a contradiction. ∎

Step 3 *There does not exist an $x \in (I(u,v) \setminus (I(u,\alpha) \cup I(\bar{\alpha},v)))$ of level i such that it has a neighbor of level $i-1$ in $I(u,\alpha)$, and one of level $i+1$ in $I(\bar{\alpha},v)$.*

Proof.
Suppose that there exists such an x and let its neighbors be z and w ($l(z) = i-1$ and $l(w) = i+1$). Then $d(z,w) = 2$, so $I(z,w) \cong \mathcal{B}_2$. Hence, there exists a vertex t such that $t \in I(z,w) \setminus \{x\}$. It is easy to see that t is of level i in $I(u,v)$. Suppose first, that $w \neq v$ and $z \neq u$. If $t \notin I(\bar{\alpha},v)$, then w would have at least $i+2$ neighbors of level i in $I(u,v)$ that contradicts to the induction hypothesis $I(u,w) \bowtie \mathcal{B}_{i+1}$. On the other hand, if $t \in I(\bar{\alpha},v)$, then z has at least $d-i+2$ neighbors on level i. If $z \neq u$, then this contradicts to the induction hypothesis. Now let $w = v$. Pick a neighbor q of v in $I(\bar{\alpha},v)$. Let a be a common neighbor of x and q on level $d-2$, and let b be that of α and q provided by the quadrangle property. Then $q \notin I(u,\alpha) \cup I(\bar{\alpha},v)$ and $b \in I(u,\alpha)$. So q has at least d neighbors on level $d-2$ in $I(u,v)$ that contradicts to the induction hypothesis $I(u,q) \bowtie \mathcal{B}_{d-1}$. For $z = u$ we obtain contradiction in similar way as in the case $w = v$. ∎

Step 4 *If $l(y) = j \neq d$ and all neighbors of y of level $j-1$ are in $I(u,v) \setminus I(u,\alpha)$, then $I(u,\alpha) \cap I(u,y) = \{u\}$.*

Proof.
Suppose that $z \neq u$, such that $z \in I(u, \alpha) \cap I(u, y)$ and z is of the highest level subject to this condition. Let $l(z) = k$. Then z has $d - 1 - k$ neighbors of level $k + 1$ in $I(u, \alpha)$. Furthermore, z has $j - k$ neighbors of level $k + 1$ in $I(u, y)$. All these are different, so z has at least $d + j - 2k - 1 > d - k$ neighbors of level $k + 1$ in $I(u, v)$ that contradicts to the induction hypothesis $I(z, v) \bowtie \mathcal{B}_{d-k}$. ∎

Note 1 *Similar proposition holds if we consider level $j + 1$ neighbors with the conclusion that $I(\bar{\alpha}, v) \cap I(y, v) = \{v\}$.*

Step 5 *If $z \in I(u, \alpha)$, $z \neq u$, $l(z) = i$, $w \in I(u, v) \setminus I(u, \alpha)$, $l(w) = i + 1$ and $\{z, w\}$ is an edge, then $w \in I(\bar{\alpha}, v)$. The same holds for changing $I(u, \alpha)$ to $I(\bar{\alpha}, v)$ and $i + 1$ to $i - 1$.*

Proof.
Suppose that $w \notin I(\bar{\alpha}, v)$. Then no neighbor of w of level $i + 2$ can be in $I(\bar{\alpha}, v)$ by Step 3. Thus, by Step 4 we have that $I(\bar{\alpha}, v) \cap I(w, v) = \{v\}$. Let $w = t_0, t_1, \ldots, t_s = v$ be a shortest ($d - i - 1$-long) path from w to v. Because $I(u, t_{s-1}) \cap I(u, \alpha) \ni z \neq u$, t_{s-1} must have a neighbor of level $d - 2$ in $I(u, \alpha)$. This contradicts to Step 3. ∎

Note 2 *If $z \in I(u, \alpha)$ ($z \in I(\bar{\alpha}, v)$) such that $l(z) = i$, then all neighbors of z of level $i - 1$ ($i + 1$) are in $I(u, \alpha)$ ($I(\bar{\alpha}, v)$).*

Step 6 *For every $z \in I(u, \alpha)$, such that $z \neq u$ and $l(z) = i$, there exists exactly one w $l(w) = i + 1$ such that $\{z, w\}$ is an edge and $w \notin I(u, \alpha)$. The same holds for exchanging $I(u, \alpha)$ to $I(\bar{\alpha}, v)$ and $i + 1$ to $i - 1$.*

Proof.
$I(z, v) \bowtie \mathcal{B}_{d-i}$ by the induction hypothesis ($z \neq u$), so z has $d - i$ neighbors of level $i + 1$ in $I(u, v)$. Out of those $d - i - 1$ lie in $I(u, \alpha)$. ∎

Let us denote the above w by z' and let $u' = \bar{\alpha}$ by definition. Then by Step 3-5, $z' \in I(\bar{\alpha}, v)$ if $z \in I(u, \alpha)$ and vice versa. Furthermore, $(z')' = z$. Hence, $z \mapsto z'$ is a bijection between $I(u, \alpha)$ and $I(\bar{\alpha}, v)$.

Step 7 *$z \mapsto z'$ is an isomorphism between the graphs induced by edges between different levels of $I(u, \alpha)$ and $I(\bar{\alpha}, v)$, respectively.*

Proof.
It is enough to show that $\{x, y\}$ is an edge in $I(u, \alpha)$ implies $\{x', y'\}$ is an edge in $I(\bar{\alpha}, v)$. The reverse comes from $(z')' = z$. Let $\{x, y\}$ be an edge, such that $l(x) = i$ and $l(y) = i+1$. Then $d(x, y') = d(x', y) = 2$, so x and y' must have a unique common neighbor other than y. If it is in $I(\bar{\alpha}, v)$, then by Step 6, it must be x'. If it is in $I(u, \alpha)$, then it must be on level $i + 1$. Then again by Step 6, it must be y, a contradiction. If it is in $I(u, v) \setminus (I(u, \alpha) \cup I(\bar{\alpha}, v))$, then it contradicts to Step 3. ∎

Step 8 *If $l(z) = i$, $z \in I(u, \alpha)$, $l(w) \leq i$ and $w \in I(\bar{\alpha}, v)$, then $\{z, w\}$ is not an edge.*

Proof.
By the triangle inequality we have that

$$\begin{aligned} d(\alpha, \bar{\alpha}) &\leq d(\alpha, z) + d(z, w) + d(w, \bar{\alpha}) \\ &\leq d - 1 - i + 1 + i - 1 = d - 1 \end{aligned}$$

a contradiction. ∎

Now, if we do not consider edges between vertices of the same level, then $I(u, \alpha)$ and $I(\bar{\alpha}, v)$ are $d - 1$ dimensional hypercubes, respectively. Furthermore, the latter one is positioned one level higher, and there is an edge between corresponding vertices of $I(u, \alpha)$ and $I(\bar{\alpha}, v)$. No other edge is going between vertices of different levels so we can conclude that $I(u, \alpha) \cup I(\bar{\alpha}, v) \bowtie \mathcal{B}_d$.

Step 9 *$I(u, v) = I(u, \alpha) \cup I(\bar{\alpha}, v)$.*

Proof.
Suppose in contrary that there exists a neighbor x of v in $I(u, v) \setminus (I(u, \alpha) \cup I(\bar{\alpha}, v))$. Then by the quadrangle property there exists $z \in I(u, v)$ of level $d - 2$ such that z is connected to both x and α. This implies that $z \in I(u, \alpha)$. However, this contradicts to Step 3. ∎

Proof of Theorem 1.8.
Let $u, v \in V(G)$ and $x, y \in I(u, v)$. Let us denote the distance of x and y in the graph $I(u, v)$ by $d_{uv}(x, y)$ to distinguish from their distance in G, $d(x, y)$. If $d_{uv}(x, y) = d(x, y) = b$, then $I(x, y) \bowtie \mathcal{B}_b$. However, $I(u, v)$ contains a b-cube between x and y because $I(u, v) \bowtie \mathcal{B}_d$. This means that $I(x, y) \subset I(u, v)$. On the other hand, if $d_{uv}(x, y) > d(x, y)$, then x does not have a diametrical pair in $I(u, v)$, which contradicts to the assumption that G is spherical. ∎

References

[HL] I. Havel and P. Liebl (Prague), unpublished.

[M] H.M. Mulder, *The Interval Function of a Graph*, Math. Centre Tracts, **132** Amsterdam (1980).