# Edge Isoperimetric Problems on Graphs\*

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#### Abstract

We survey results on edge isoperimetric problems on graphs, present some new results and show some applications of such problems in combinatorics and computer science.

# 1 Introduction

Let  $G = (V_G, E_G)$  be a simple connected graph. For a subset  $A \subseteq V_G$  denote

$$I_G(A) = \{(u, v) \in E_G \mid u, v \in A\},\$$
  
$$\theta_G(A) = \{(u, v) \in E_G \mid u \in A, v \notin A\}.$$

We omit the subscript G if the graph is uniquely defined by the context. By edge isoperimetric problems we mean the problem of estimation of the maximum and minimum of the functions I and  $\theta$  respectively, taken over all subsets of  $V_G$  of the same cardinality. The subsets on which the extremal values of I (or  $\theta$ ) are attained are called *isoperimetric subsets*.

These problems are discrete analogies of some continuous problems, many of which can be found in the book of Pólya and Szegö [99] devoted to continuous isoperimetric inequalities and their numerous applications. Although the continuous isoperimetric problems have a history of thousand years, the discrete structures studied mostly in the present century, also gave rise to hundreds of specific discrete problems. Two of such problems we study in our paper.

Denote

$$I_G(m) = \max_{\substack{A \subseteq V_G \\ |A|=m}} |I_G(A)|, \qquad \theta_G(m) = \min_{\substack{A \subseteq V_G \\ |A|=m}} |\theta_G(A)|.$$

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The two discrete problems mentioned above are closely related and for k-regular graphs are equivalent due to the equation

$$2 \cdot |I_G(A)| + |\theta_G(A)| = k \cdot |A|, \tag{1}$$

which implies  $2I_G(m) + \theta_G(m) = km$ ,  $m = 1, \ldots, |V_G|$ . For non-regular graphs, however, the difference between the two problems can be significant, as we will demonstrate below.

The both problems are known to be NP-hard in general [57], however the problem of maximization of I is a bit simpler in a sense, because it is free of so-called "border effects". Consider for example a two-dimensional grid and let m = 4. It is a simple exercise to show that a cycle of length 4 provides an isoperimetric set with respect to the function I and for each such a cycle the value of I is the same. Also if each side of the grid consists of at least 4 vertices, the same cycle located in one of the grid corners provides minimum to the function  $\theta$ . However, in this case the value of  $\theta$  of a 4-cycle strictly depends on the location of the cycle in the grid. Because of such effects most of the exact results concern the maximization problem, however in most applications the function  $\theta$  arises.

We distinguish three kinds of solutions of our problems. The general goal is to find a function f(m, G) such that  $\theta_G(m) \ge f(m, G)$ . We call such inequality isoperimetric inequality. Ideally, one would determine the function  $\theta_G(m)$  in order to get the best possible inequality of this kind. The next step is to specify for each m one of the isoperimetric subsets. Usually, the structure of an isoperimetric set allows to find the corresponding function  $I_G(m)$  or  $\theta_G(m)$ , which may look horrible and for a number of applications a lower bound on  $\theta$  is often preferable. Finally, the last step is to specify for a given m all isoperimetric subsets of this cardinality.

In the most of known in the literature cases to specify one of the isoperimetric subsets constructively is possible if the problem has the nested solutions property. By this we mean the existence of isoperimetric (with respect to I or  $\theta$ ) subsets  $A_i \subseteq V_G$ ,  $i = 1, \ldots, |V_G|$ , with  $|A_i| = i$ such that

$$A_1 \subset A_2 \subset \cdots \subset A_{|V_G|}.$$

In other words, there exists an order of vertices of G induced by the inclusions above, such that each *initial segment* of this order represents an isoperimetric subset with respect to the considered function. We call such an order *optimal order*. However, in a number of cases isoperimetric inequalities are strict for particular values of m and provide isoperimetric subsets even if the nested solutions property does not exist.

The paper is organized as follows. The next Section 2 is devoted to the proof technique of exact results on the edge isoperimetric problem for a number of graphs and to some methods of obtaining isoperimetric inequalities. In Section 3, we extend the methods of Section 2 for maximization of more general functions and present a kind of equivalence relations allowing to solve an edge isoperimetric problem for some classes of graphs if it is solved for a representative of this class. In Section 4, we list some applications of the edge-isoperimetric problems and conclude the paper with Section 5 containing some remarks and research topics.

Note that there, of course, exist vertex-isoperimetric problems consisting of minimization of e.g. the number of vertices on distance 1 of a set. We refer the reader to the survey paper [16] or to the book [38] devoted to such problems and their applications.

# 2 Proof technique

In this section, we stress our attention on some combinatorial methods for construction of isoperimetric subsets, some methods of continuous approximation of the considered discrete problems providing good isoperimetric inequalities and on methods involving eigenvalues of related matrices and isoperimetric constants.

## 2.1 Combinatorial results

Let us start with simple graphs first.

**Proposition 2.1** Let T be a tree with p vertices. Then I(m) = m - 1.

Clearly, if  $A \subseteq V_T$ , then the number of edges in the induced by the vertex set A subgraph G' equals |A| - c(G'), where c(G') is the number of components of G'. Therefore the maximization of the function I is equivalent is this case to minimization of the number of components. Clearly, if a vertex set A induces a connected component and there exists a vertex  $a \in V_T \setminus A$ , then the vertex set  $A \cup \{a\}$  also induces a connected component. Therefore, the problem of maximization of I for trees has the nested solutions property and there is a simple way to generate all optimal numberings.

Although minimization of  $\theta$  is trivial for the clique, this problem for a complete *n*-partite graph is not [94]. The complete *n*-partite graph  $K_{p_1,\ldots,p_n}$  is defined as a graph, whose vertex set can be partitioned into *n* subsets  $P_1,\ldots,P_n$  (independent subsets) so that two vertices are adjacent iff they belong to different  $P_i$ . Denote  $p_i = |P_i|$  and  $p = p_1 + \cdots + p_n$  and assume that  $p_1 \ge p_2 \ge \cdots \ge p_n$ . Consider the following numbering  $\mathcal{K}$  of the vertex set of  $K_{p_1,\ldots,p_n}$  by numbers  $1, 2, \ldots, p$ . The numbering process consists of repetition of the following procedure, each time with the next numbers in increasing and decreasing order respectively until all the vertices are numbered:

Let k be maximal integer for which  $p_1 = p_2 = \cdots = p_k$ . Take one (arbitrary) vertex from each set  $P_i$  (i = 1, 2, ..., k) and number them by 1, ..., k in arbitrary order. Repeat the same operation: take one (arbitrary) vertex from each set  $P_i$  (i = 1, 2, ..., k) but number them by p, ..., p - k + 1 in arbitrary order. Remove the 2k numbered vertices from the graph together with all incident edges, obtaining this way a new complete multipartite graph. Reorder (if necessary) the independent sets of this graph placing them in decreasing order of their cardinalities.

Fig. 1 shows the numbering of  $K_{5,5,3,2}$ . In this picture the independent sets of the graph are shown in ovals and all the edges are omitted.

#### Theorem 2.1 (Muradyan [94])

For each m, m = 1, ..., p, the collection of the first m vertices of  $K_{p_1,...,p_n}$  taken in the order  $\mathcal{K}$  provides minimum for the function  $\theta$ .

$$\underbrace{\begin{array}{c} \bullet \bullet \bullet \bullet \bullet \\ 1 & 15 & 3 & 13 & 7 \end{array}}_{2 & 14 & 4 & 12 & 8 \\ \end{array}} \underbrace{\begin{array}{c} \bullet \bullet \bullet \\ 5 & 11 & 9 \\ \end{array}}_{5 & 11 & 9 \\ \end{array}} \underbrace{\begin{array}{c} \bullet \bullet \\ 10 & 6 \\ \end{array}}_{10 & 6 \\ \end{array}}$$

Figure 1: Numbering of vertices of  $K_{5,5,3,2}$ 

These results are extendible for cartesian products of such graphs, and for this operation a nice technique is designed based on the notion of compression and stabilization. By the cartesian product  $G_1 \times \cdots \times G_n$  of  $n \ge 2$  graphs  $G_i = (V_i, E_i), i = 1, \ldots, n$ , we mean the graph on the vertex set  $V_1 \times \cdots \times V_n$  where two tuples  $(v_1, \ldots, v_n)$  and  $(u_1, \ldots, u_n)$  form an edge iff they agree in some n-1 entries and for the remaining entry, say i, holds  $(v_i, u_i) \in E_i$ .

One of the first classical results on edge isoperimetric problems for the cartesian products was proved by Harper [63] for the binary *n*-cube  $Q^n$ . In order to present his and some other results we need the notion of the *lexicographic order*  $\mathcal{L}$ . Let  $[k] = \{0, \ldots, k-1\}$  and consider two vectors  $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [k_1] \times [k_2] \times \cdots \times [k_n]$ . We say **x** is less than **y** in the lexicographic order (notation  $\mathbf{x} <_{\mathcal{L}} \mathbf{y}$ ) iff there exists an index *i* such that  $x_1 = y_1, \ldots, x_{i-1} = y_{i-1}$  and  $x_i < y_i$  holds. Denote by  $\mathcal{L}(m)$  the collection of the first *m* vectors of  $[k_1] \times \cdots \times [k_n]$  taken in the lexicographic order.

#### **Theorem 2.2** (Harper [64])

For any subset  $A \subseteq V_{Q^n}$  it holds  $|I(A)| \leq |I(\mathcal{L}(|A|))|$ , where the minimum is attained only for the sets A isometrically equivalent to  $\mathcal{L}(|A|)$ .

Two subsets  $A, B \subseteq V_{Q^n}$  are called isometrically equivalent if  $B = \varphi(A \oplus \mathbf{a})$  for some  $\mathbf{a} = (a_1, \ldots, a_n) \in V_{Q^n}$ . Here by  $A \oplus \mathbf{a}$  we denote the subset obtained by inverting those positions  $x_i$  of all vectors of A for which  $a_i = 1$  holds. Furthermore, for a permutation  $\phi \in \mathbf{S}_n$  we denote by  $\phi(A)$  subset obtained from A by applying  $\phi$  to coordinates of all its vectors. Clearly, none of these transformations changes the size of I(A) or  $\theta(A)$ .

Theorem 2.2 was rediscovered a number of times (cf. e.g. [12, 62]). Many applications and with years development of the extremal set theory lead to a very simple proof, which now may be considered as a standard and powerful approach working for many other problems.

We denote  $A_{\sigma}(i) = \{(a_1, \ldots, a_n) \in A \mid a_i = \sigma\}$  for  $\sigma \in \{0, 1\}$  and let  $m_{\sigma} = |A_{\sigma}(i)|$ . Introduce the compression operator  $C_i(A)$ , which replaces the set  $A_{\sigma}(i)$  with the first  $m_{\sigma}$  vertices of the subcube  $x_i = \sigma$  taken in the lexicographic order. The proof is done by induction on n. Using the induction hypothesis it is easy to verify

**Lemma 2.1** It holds:  $|I(C_i(A))| \ge |I(A)|$ .

Now let us apply the operator  $C_i$  for i = 1, 2, ..., n. The crucial observation is that after a finite number of such transformations we get a subset  $B \subseteq V_{Q^n}$  which is stable under the compression, i.e.,  $C_i(B) = B$  for i = 1, ..., n. It turns out that the structure of a stable set is almost well defined:

Lemma 2.2 If  $C_i(B) = B$  for i = 1, ..., n, then either  $B = \mathcal{L}(|B|)$  or  $|B| = 2^{n-1}$  and  $B = \mathcal{L}(2^{n-1} - 1) \cup \{(1, 0, ..., 0)\}.$ 

To complete the proof it is left to verify that in the case  $|B| = 2^{n-1}$  the first alternative provided by Lemma 2.2 is strictly better. With a little stricter analysis one can also prove the uniqueness.

The result of Harper was later extended in several directions. Let us first consider the Hamming graph  $H(p_1, \ldots, p_n)$ , i.e., the cartesian product of complete graphs  $K_{p_1} \times \cdots \times K_{p_n}$  and let  $p_1 \leq \cdots \leq p_n$ .

#### **Theorem 2.3** (Lindsey [83])

For each  $m, m = 1, ..., p_1 \cdots p_n$ , the collection of the first m vertices of  $H(p_1, ..., p_n)$  taken in the lexicographic order provides maximum for the function I.

This theorem one can also find in some other papers [75, 46, 84] and it can be proved with the compression approach above. The proof starts with verifying the inductive hypothesis for n = 2. For  $n \ge 3$  we use representation  $H(p_1, \ldots, p_n) = H(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n) \times K_{p_i}$ for some *i* and make the compression in all components  $H(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$ . After applying the similar stabilization arguments we come to a stable set of the same cardinality with no less value of *I*.

Unfortunately, in the case of the Hamming graph the structure of a stable set B is not so simple. Let us denote by **b** the last vertex of B in the lexicographic order and by **a** the first vertex of  $H(p_1, \ldots, p_n) \setminus B$  in the same order. Clearly, if  $\mathbf{b} <_{\mathcal{L}} \mathbf{a}$ , the proof is finished. Otherwise it is remained to show that the exchange of **b** and **a** does not decrease the function I, which leads to consideration of a number of cases. It should be mentioned that for Hamming graphs the structure of stable sets is also known (see [4, 27]), but it does not make the proof much shorter. It is easy to verify that the optimal subsets can also have another form (cf. [75]).

The next extension concerns the cartesian product of complete bipartite graphs with the independent sets of the same size. Let  $p_1 \leq \cdots \leq p_n$  and  $F(p_1, \ldots, p_n) = K_{p_1,p_1} \times \cdots \times K_{p_n,p_n}$ . The result is quite similar to the results of Harper and Lindsey:

Theorem 2.4 (Ahlswede, Cai [4])

For each  $m, m = 1, ..., p_1 \cdots p_n$ , the collection of the first m vertices of  $F(p_1, ..., p_n)$  taken in the lexicographic order provides maximum for the function I.

Here the proof also claims to consider the case n = 2 separately, but, however, it is based on an interesting observation, called by the authors "the local-global principle" which we consider in Section 3.2. It is interesting that the nested solution property does not take place in general if the independent sets of the bipartite graphs are of different size. This result is in turn extended in [21] in various directions. Thus, it is proved there that the lexicographic order still works well for the powers of complete *t*-partite graphs  $K_{p,p,\ldots,p}$ . The graphs in the product should be numbered as explained in Fig. 1.

New types of graphs are introduced in [21]. Consider the complete bipartite graph  $K_{p,p}$  and remove a perfect matching from it. The resulting graph is regular and has a perfect matching as well. Removal t perfect matchings from  $K_{p,p}$  results in a regular graph which we denote by B(p,t). The graphs obtained this way are not isomorphic but as it is easily shown, they all have the same function I.

Furthermore, denote by H(p,t) the graph obtained from B(p,t) by joining any pair of vertices within each independent set (of size p). The graph H(p,t) is regular of degree 2p - t - 1 and is of interest due to the following inequality. If G is a regular graph of the order 2p and degree 2p-t-1 and with the nested solutions property in the edge-isoperimetric problem, then for the cartesian product of n such graphs it holds  $I_{H(p,t)\times\cdots\times H(p,t)}(m) \geq I_{G\times\cdots\times G}(m), m = 1, \ldots, (2n)^n$ .

Theorem 2.5 (Bezrukov, Elsässer [21])

For  $t \in \{0, \ldots, \lceil p/2 \rceil, p-1\}$  and each  $m, m = 1, \ldots, (2p)^n$ , the collection of the first m vertices of  $B(p,t) \times \cdots \times B(p,t)$  (n times) (resp. of  $H(p,t) \times \cdots \times H(p,t)$ ) taken in the lexicographic order provides maximum for the function I.

Thus, Theorem 2.5 provides a best possible isoperimetric inequality for products of regular graphs. We conjecture that this theorem concerning the graphs B(p,t) also provides such inequality for products of regular bipartite graphs. It is interesting to mention that for any t outside the range mentioned in the theorem the products of graphs B(p,t) and H(p,t) do not have nested solutions property even for n = 2.

Since the last considered graphs are regular, then the same sets automatically provide a solution for the minimization of  $\theta$ . However, this is not the case with the grid  $G(p_1, \ldots, p_n)$  defined as the cartesian product of paths with  $p_1, \ldots, p_n$  vertices. We assume that  $p_1 \leq \cdots \leq p_n$  and introduce the following order  $\mathcal{I}$ .

For a vector  $\mathbf{x} = (x_1, \ldots, x_n)$  denote  $]\mathbf{x}[= \max_i x_i$  and let  $\tilde{\mathbf{x}}$  be the vector obtained from  $\mathbf{x}$  by replacing all entries not equal to  $]\mathbf{x}[$  by 0. The order  $\mathcal{I}$  is defined inductively. For  $\mathbf{x}, \mathbf{y}$  we say  $\mathbf{x} >_{\mathcal{I}} \mathbf{y}$  iff

- a.  $\mathbf{x} > \mathbf{y}$ , or
- b.  $]\mathbf{x}[=]\mathbf{y}[$  and  $\tilde{\mathbf{x}} >_{\mathcal{L}} \tilde{\mathbf{y}},$  or
- c.  $]\mathbf{x}[=]\mathbf{y}[=t > 1, \ \tilde{\mathbf{x}} = \tilde{\mathbf{y}}, \ \text{and} \ \mathbf{x}' >_{\mathcal{I}} \mathbf{y}', \ \text{where} \ \mathbf{x}', \mathbf{y}' \ \text{are obtained from } \mathbf{x}, \mathbf{y} \ \text{respectively by} \ \text{deleting all entries with} \ x_i = y_i = t, \ \text{and} \ \mathcal{L} \ \text{is the lexicographic order.}$

**Theorem 2.6** (Bollobás, Leader [32] for  $p_1 = p_n$ , Ahlswede, Bezrukov [1] in general) For each  $m, m = 1, \ldots, p_1 \cdots p_n$ , the collection of the first m vertices of  $G(p_1, \ldots, p_n)$  taken in the order  $\mathcal{I}$  provides maximum for the function I. In the two-dimensional case taking the first  $m = r^2$  vertices of this order for some  $r \leq p_1$  we get a quad  $\{(x, y) \mid x, y < r\}$  located in a corner of the grid. This corresponds to the well-known fact of the Euclidean geometry that a quad with a given perimeter has maximal square in the class of rectilinear figures on the plane. The proof in general is based on the compression and stabilization arguments as well, followed by exchange of appropriate vertices as in the case of the Hamming graphs.

Several results are known for two-dimensional (infinite) grid-like graphs. The vertex sets of these graphs are given by  $\{(x, y) \mid x, y \text{ are integers}\}$ . The edge sets of the graphs  $G_1, G_2$  and  $G_3$  are defined as

$$E_{G_1} = \{ ((x, y), (x', y')) \mid |x - x'| + |y - y'| = 1 \}, E_{G_2} = E_{G_1} \cup \{ ((x, y), (x', y')) \mid x' = x + 1, y' = y + 1 \}, E_{G_3} = E_{G_2} \cup \{ ((x, y), (x', y')) \mid x' = x + 1, y' = y - 1 \}.$$

Thus informally the graph  $G_1$  is simply an infinite grid,  $G_2$  is a grid with one diagonal in each cell (i.e. a cycle of length 4) and  $G_3$  is a grid with two diagonals in each cell. It is shown (cf. [60, 61, 36] respectively) that

$$I_{G_1}(m) = \left\lfloor 2m - 2\sqrt{m} \right\rfloor,$$
  

$$I_{G_2}(m) = \left\lfloor 3m - \sqrt{12m - 3} \right\rfloor,$$
  

$$I_{G_3}(m) = \left\lfloor 4m - \sqrt{28m - 12} \right\rfloor$$

The corresponding extremal sets of m vertices grow with m like a spiral. The graph  $G_3$  can be viewed as the Shannon product of paths. See [4] for asymptotically optimal bounds for the Shannon product of arbitrary graphs.

The problem of minimization of  $\theta$  for the grid is much more difficult. However, if all the  $p_i$ 's are infinite, then the problem still has the nested solutions property and the same order  $\mathcal{I}$  provides isoperimetric subsets with respect to  $\theta$  [1]. If all the  $p_i$ 's are finite, then this problem does not have the nested solutions property even for n = 2. In the two-dimensional case the exact solution is easy to get (cf. [1, 32]), however in the general case the exact solution is known just for some special cardinalities, which follows from the isoperimetric inequalities below.

A more complicated order provides nestedness for the cartesian powers  $P^n$  of the Petersen graph P [24]. The orders  $\mathcal{P}^1$  and  $\mathcal{P}^2$  are shown in Fig. 2. The vertices of the graph  $P^2 = P \times P$ are represented as the entries of a 10 × 10 matrix  $\{a_{i,j}\}$ , where i, j = 0, ..., 9. It is assumed in this figure that the entry  $a_{0,0}$  is in the bottom left corner of the matrix. Furthermore, it is assumed that the elements  $a_{0,0}, ..., a_{9,0}$  of the bottom row and the elements  $a_{0,0}, ..., a_{0,9}$  of the leftmost column represent the vertices of the multiplicands of the product (i.e., vertices of P) taken in the order  $\mathcal{P}^1$ . The value of the matrix element  $a_{i,j}$  is the number of the corresponding vertex of the graph  $P^2$  in the order  $\mathcal{P}^2$ , as shown in Fig. 2b).

With the help of a computer it is verified in [24] that any initial segment of the order  $\mathcal{P}^2$  is an optimal set. To verify this one can consider compressed sets only. In this case there are  $\binom{20}{10} = 352,716$  compressed sets. The complete choice of such a size is doable by computer but without compression there are  $2^{100} \approx 1.3 \times 10^{30}$  possibilities, a prohibitively large number.



Figure 2: The orders  $\mathcal{P}^1$  (a) and  $\mathcal{P}^2$  (b)

For  $n \geq 2$  and  $x_i, y_i \in \{0, \ldots, 9\}$  we say that  $(x_1, \ldots, x_n) >_{\mathcal{P}^n} (y_1, \ldots, y_n)$  iff

- a.  $x_1 1 > y_1$ , or
- b.  $x_1 1 = y_1$  and  $y_1 \in \{1, 2, 4, 6, 7\}$ , or c.  $x_1 - 1 = y_1, y_1 \in \{0, 3, 5, 8\}$  and  $(x_2, \dots, x_n) \ge_{\mathcal{P}^{n-1}} (y_2, \dots, y_n)$ , or d.  $x_1 = y_1$  and  $(x_2, \dots, x_n) >_{\mathcal{P}^{n-1}} (y_2, \dots, y_n)$ , or e.  $x_1 + 1 = y_1, y_1 \in \{1, 4, 6, 9\}$  and  $(x_2, \dots, x_n) >_{\mathcal{P}^{n-1}} (y_2, \dots, y_n)$ .

**Theorem 2.7** (Bezrukov, Das, Elsässer [24]) Any initial segment of the order  $\mathcal{P}^n$  for  $n \ge 2$  is an optimal set.

Let us mention one more result where the compression technique is effectively applied. Given two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , denote by  $G_1[G_2]$  their composition, i.e., the graph on the vertex set  $V_1 \times V_2$  where two vertices  $((v_1, u_1), (v_2, u_2))$  are adjacent iff  $(v_1, v_2) \in E_1$  or  $v_1 = v_2$  and  $(u_1, u_2) \in E_2$ . Let  $P_q$  and  $C_q$  denote the path and the circle with q vertices respectively.

**Theorem 2.8** (Liu, Williams [85]) If  $p \leq q$  and  $G = K_p[P_q]$  or  $G = K_p[C_q]$ , then for each m, m = 1, ..., pq, the collection of the first m vertices of G taken in the lexicographic order provides minimum for the function  $\theta$ .

The proof is based on the compression approach used in the paper of Chvátalová [45] who studied the bandwidth of two-dimensional grids.

### 2.2 Isoperimetric inequalities

First consider the *n*-cube  $Q^n$ . For each integer  $m, 1 \le m \le 2^n$ , there exist integers  $a_1, \ldots, a_t$  with  $n \ge a_1 > a_2 > \cdots > a_t \ge 0$ , such that

$$m = 2^{a_1} + 2^{a_2} + \dots + 2^{a_t},$$

and this representation is unique. By induction it is easy to show that

$$I_{Q^n}(m) = \left[a_1 \cdot 2^{a_1 - 1} + \dots + a_t \cdot 2^{a_t - 1}\right] + \left[2^{a_2} + 2 \cdot 2^{a_3} + \dots + (t - 1) \cdot 2^{a_t}\right].$$

With the help of (1) one can also get an exact formula for  $\theta_{Q^n}(m)$ . For the applications, however, the following estimation is much more convenient:

**Theorem 2.9** (Chung, Füredi, Graham, Seymour [43]) Let m > 0. Then  $\theta_{Q^n}(m) \ge m(n - \log_2 m)$ .

Theorem 2.2 implies that this bound is strict if  $m = 2^t$ .

Concerning the minimization of  $\theta$  on grids, it is natural to consider the rectilinear bodies of the continuous *n*-dimensional cube with the side length 1. By the rectilinear body we mean a finite union of the sets of the form  $\prod_{i=1}^{n} [a_i, b_i]$ , where  $[a_i, b_i]$  with  $a_i < b_i$  is a segment. For a rectilinear body A the continuous analog of the function  $\theta$  is the surface area of A, which is expressed as an easy computable integral (cf. [32] for details).

Using the compression approach and a one-dimensional parameterization, the problem of minimization of the surface area of the rectilinear bodies can be exactly solved due to the convexity of the functions involved. The obtained solution of the continuous problem provides a lower bound for its discrete version:

**Theorem 2.10** (Bollobás, Leader [32]) Let  $m \leq p^n$ . Then  $\theta_{G(p,\ldots,p)}(m) \geq \min_{1 \leq k \leq n} \left\{ m^{1-1/k} k p^{n/k-1} \right\}$ .

This result shows that for m of the form  $m = t^k p^{n-k}$  an  $\theta$ -optimal set is among the sets of the form  $\{[t]^k \times [p]^{n-k}\}, k = 1, ..., n$ . Therefore, the problem of minimization of  $\theta$  does not have the nested solutions property.

Finding of continuous analogs also helps in analyzing the Kleitman-West problem. Denote

$$Q_k^n = \{ (a_1, \dots, a_n) \in V_{Q^n} \mid a_1 + \dots + a_n = k \}$$

and define the graph J(n, k) on the vertex set  $Q_k^n$  by joining with an edge each two vertices of  $Q_k^n$  on Hamming distance 2. Thus, the graph J(n, k) is regular. The case k = 2 was studied by Ahlswede and Katona [5], who proved that an isoperimetric set is formed either by the first or the last vertices of  $Q_2^n$  taken in the lexicographic order.

Harper in [67] and Ahlswede and Cai in [3] transformed the problem of minimization of  $\theta$  on the graph J(n,k) into the problem of maximizing some functions defined on downsets of the poset L(n-k,k). The poset L(n-k,k) is defined by the integer sequences  $(a_1,\ldots,a_k)$  with  $0 \le a_1 \le \cdots \le a_k \le n-k$  ordered coordinatewise. We call a subset  $S \subseteq L(n-k,k)$  downset if the conditions  $a \in S$  and  $b \prec a$  imply  $b \in S$ . Denoting  $r(m) = \max_{|S|=m} \sum_{a \in S} (a_1 + \cdots + a_k)$ taken over all downsets S, one has similarly to (1):  $\theta(m) = k(n-k) - 2r(m)$ .

Such a transformation makes the proof in the case k = 2 much shorter comparing to [5] due to a simple geometric interpretation of L(n-2, 2). In [3] the authors mostly studied the case k = 3 using essentially combinatorial arguments and obtained an exact solution of the continuous problem for this case.

In a continuous analog of this problem we let  $n \to \infty$  and obtain the set  $L_k = \{(x_1, \ldots, x_k) \in \mathbf{R}^k \mid 0 \leq x_1 \leq \cdots \leq x_k \leq 1\}$  ordered coordinatewise. Since  $\int_{L_k} dx = 1/k!$ , the problem now is for a given  $l \leq 1/k!$  to maximize  $r(S) = \int_S r(\mathbf{x}) dx$  over all downsets  $S \subseteq L_k$  with volume l, where  $r(\mathbf{x}) = \sum_{i=1}^k x_i$ . Denote this maximal value by r'(l). Harper in [67] used variational methods to show that only the sets  $S_j = \{x \in L_k \mid x_j \leq t\}$  can be optimal, where t is chosen so that the volume of  $S_j$  is l. Furthermore, he also proved that the optimum is attained only either for j = 1 or j = k.

# **Theorem 2.11** (Harper [67])

Let for a given  $l, l \leq 1/k!$ , t be defined by

$$\int_{S_j(t)} \mathrm{d}x = \frac{1}{k!} \sum_{i \ge j} \binom{k}{i} t^i (1-t)^{k-i} = l.$$

Then it holds

$$r'(l) = \min_{j \in \{1,k\}} \int_{S_j(t)} r(\mathbf{x}) \, \mathrm{d}x = \frac{1}{k!} \sum_{i \ge j} (kt + k - i) \binom{k}{i} t^i (1-t)^{k-i}.$$

The isoperimetric sets  $S_j(t)$  of the continuous model provide corresponding extremal downsets for the discrete model in the case of "good" cardinalities m of the form  $m = \binom{n}{k} - \binom{a}{k}, 0 \le a \le n$ . It is, however, not clear what happens in between these good cardinalities. Counterexamples to a natural conjecture in [5] were constructed in the discrete case k = 3 in [3] (see also [67] for more details).

It was noticed by many researchers that eigenvalues of graphs play an important role in combinatorial optimization (see the survey paper [91] for a number of applications). Let  $A(G) = \{a_{ij}\}$ be the adjacency matrix of a graph G and denote by L(G) its Laplacian matrix obtained from A(G) by multiplying all its elements with -1 and replacing  $a_{ii} = 0$  with the degree of the  $i^{th}$ vertex of G. It is well known that L(G) is positive semidefinite, all eigenvalues of L(G) are real and non-negative, the smallest eigenvalue is equal to 0 and the second smallest eigenvalue  $\lambda_G$ is positive iff G is connected. It is also known that  $\lambda_G$  can be estimated within arbitrary given accuracy in polynomial time. Let  $|V_G| = p$  and  $f : V_G \mapsto \mathbf{R}$  be some mapping. Using the Lagrange identity, Fiedler [54] rewrote the Courant-Fisher principle in the form

$$\lambda_G = \min_{f \neq 0} \frac{2n \cdot \sum_{(u,v) \in E_G} (f(u) - f(v))^2}{\sum_{u \in V_G} \sum_{v \in V_G} (f(u) - f(v))^2}.$$
(2)

The minimum is attained for any eigenvector corresponding to  $\lambda_G$ . Now, by taking  $A \subseteq V_G$  with |A| = m and  $\theta_G(A) = \theta_G(m)$  and substituting to (2) the characteristic function of the set A defined by f(u) = 1 for  $u \in A$  and f(u) = 0 for  $u \notin A$  one immediately gets the following result, which is a weaker version of result in [8]:

**Theorem 2.12** (Alon, Milman [8]) For any graph G with p vertices holds

$$\theta_G(m) \ge \lambda_G \frac{m(p-m)}{p}.$$

This bound is attainable for complete graphs and is the case m = p/2 for the *n*-cube  $Q^n$   $(\lambda_{Q^n} = 2)$  and the Petersen graph P shown in Fig. 4a  $(\lambda_P = 2)$ . In the last case we make a partition cutting the edges connecting the inner and the outer cycles.

Since for any graphs G, H it holds  $\lambda_{G \times H} = \min\{\lambda_G, \lambda_H\}$ , the bound of Theorem 2.12 (with m = p/2) is also attainable on  $G \times H$ , with  $\lambda_G = 2$  and  $\lambda_H \ge 2$ , for example on the cartesian product of n Petersen graphs (graph  $P^n$ ) and for  $P^n \times Q^l$ . This solves a problem of Das and Öhring [50], who studied such graphs and proposed constructions for their bisection, conjecturing that they are optimal (cf. Theorem 2.7).

For a survey on spectral methods in graph theory we refer the reader to [91]. A number of further isoperimetric inequalities related to other graph parameters are stored in [89, 92]. Isoperimetric inequalities for vertex versions of the discrete isoperimetric problem can be found in [30, 31, 34, 77].

## 2.3 Graph isoperimetric constants

Closely related to the values of  $\theta_G(m)$  are the quantities which are called *isoperimetric numbers* of G. In the literature a few definitions of such numbers are known, we consider here just some of them. Assume that  $|V_G| = p$  and for  $A \subseteq V_G$  denote  $\operatorname{vol}(A) = \sum_{v \in A} \deg(v)$ . Further denote

$$i_{1}(G) = \min_{A} \frac{|\theta_{G}(A)|}{\min\{|A|, p - |A|\}},$$
  

$$i'_{1}(G) = \min_{A} \frac{|\theta_{G}(A)|}{\min\{\operatorname{vol}(A), \operatorname{vol}(V_{G} \setminus A)\}},$$
  

$$i_{2}(G) = \min_{A} \frac{|\theta_{G}(A)| \cdot p}{|A|(p - |A|)},$$
  

$$i_{3}(G) = \min_{A} \frac{|\theta_{G}(A)|}{|A|\log\frac{p}{|A|}},$$

where all the minima are running over all proper nonempty sets of  $V_G$ . Isoperimetric numbers can be used, in particular, for deriving isoperimetric inequalities as will be shown later.

Theorem 2.13 (Mohar [90] and Chung [42] respectively)

$$\lambda_G/2 \le i_1(G) \le \sqrt{\lambda_G(2\Delta_G - \lambda_G)},$$
(3)

$$\lambda'_G/2 \le i'_1(G) \le \sqrt{1 - (\lambda'_G - 1)^2}.$$
 (4)

Here  $\Delta_G$  is the maximum vertex degree and  $\lambda'_G$  is the second smallest eigenvalue of the  $p \times p$  matrix  $L'(G) = \{m_{ij}\}$  defined by  $m_{ii} = 1$ ,  $m_{ij} = (\deg(i) \cdot \deg(j))^{-1/2}$  if the vertices *i* and *j* are adjacent and  $m_{ij} = 0$  otherwise.

The lower bounds in (3) and (4) are simple and provided by a similar approach used for the proof of Theorem 2.10. The upper bounds can be considered as Cheeger-like inequalities. If the graph G is regular, then  $L'(G) = (1/\Delta_G)L(G)$  and so  $\lambda'_G = \lambda_G/\Delta_G$ . It is easily shown that the upper bound in (4) is in this case better than the one in (3). An alternative upper bound involving the genus  $g_G$  of the graph was derived by Boshier [35] and later significantly improved by Sýkora and Vrt'o [109], who showed that for  $g_G > 0$ 

$$i_1(G) \le 15\sqrt{\frac{3g_G\Delta_G}{p}}.$$

Let us refer now to the cartesian products of graphs. In the recent years this operation is extensively studied in the literature and many deep results have been found.

Theorem 2.14 (Houdré, Tetali [69], see also Chung, Tetali [44])

$$\min\{i_1(G), i_1(H)\}/2 \le i_1(G \times H) \le \min\{i_1(G), i_1(H)\}.$$
(5)

The upper bound in (5) follows immediately if one takes an isoperimetric set  $A \subseteq V_G$  (with  $|A| \leq p/2$ ) and considers the set  $B = A \times V_H \subseteq V_{G \times H}$ . For this set one has:  $|B| = |A||V_H|$  and  $|\theta(B)| = |\theta(A)||V_H|$ , thus  $i_1(G \times H) \leq |\theta(B)|/|B| = i_1(G)$ . The lower bound, however, is much more tricky and based on a modification of formula (2).

Clearly,  $i_1(K_p) = \lceil p/2 \rceil$ . In this case the following inequalities are shown in [76] as a consequence of more general results concerning graph bundles:

$$\min\{i_1(G), p/2\}/2 \le i_1(G \times K_p) \le \min\{i_1(G), \lceil p/2 \rceil\},\$$

which is an extension of the corresponding result in [90]. On the other hand [90] contains an example of graphs for which strict inequality in the upper bound in Theorem 2.14 holds. Playing with parameters of this example it is possible to construct a graph G for which  $i_1(G \times \cdots \times G) \rightarrow \ln 2 \approx 0.69$  as the number of the graphs in the product grows.

In contradistinction to the number  $i_1$  for the cartesian products, the numbers  $i_2$  and  $i_3$  behave completely different. Denote by  $G^n$  the cartesian product of n copies of a graph G.

#### **Theorem 2.15** (Tillich [111])

a. Let  $A \subseteq V_G$  be an isoperimetric set with respect to  $i_2$ . Then the set  $A \times V_{G^{n-1}}$  is isoperimetric with respect to  $i_2$ ;

b. Let  $A \subseteq V_G$  be an isoperimetric set with respect to  $i_3$ . Then for each i = 1, ..., n-1 the set  $A^i \times V_{G^{n-i}}$  is isoperimetric with respect to  $i_3$ .

More precise analysis shows [111] that

$$i_2(G \times H) = \min\{i_2(G), i_2(H)\}$$
 and  $i_3(G \times H) = \min\{i_3(G), i_3(H)\}.$ 

Observing that  $i_1(G) < i_2(G) \le 2i_1(G)$ , one gets

$$\min\{i_1(G), i_1(H)\}/2 < \min\{i_2(G), i_2(H)\}/2 \le i_1(G \times H),$$

which strengthens the lower bound in (5). It would be interesting to find examples of graphs with  $i_1(G \times H)$  that are closer to the lower bound (5).

Theorem 2.15 implies in particular that

$$i_2(G^n) = i_2(G)$$
 and  $i_3(G^n) = i_3(G)$ . (6)

Tillich in [111] found a sufficient condition for an isoperimetric constant to be "stable" with respect to cartesian products. His paper also contains an interesting observation on equivalence of certain analytic inequalities with isoperimetric inequalities for graphs.

The isoperimetric numbers may be used to derive edge isoperimetric inequalities for  $G^n$  of the form

$$\begin{array}{lll}
\theta_{G^n}(m) &\geq i_2(G) \cdot m \cdot (p^n - m)/p^n, \\
\theta_{G^n}(m) &\geq i_3(G) \cdot m \cdot \log(p^n/m).
\end{array}$$
(7)

Therefore, if we know  $i_2(G)$ , then the first lower bound in (7) is strict at least for one value of m for any n. Similarly, the knowledge of  $i_3(G)$  provides the strictness of the second inequality in (7) at least for n-1 values of m.

Consider, for example, the *n*-cube  $Q^n$ . Clearly,  $i_3(Q^1) = 1$ , thus  $i_3(Q^n) = 1$  by (6). Therefore, (7) implies  $\theta_{Q^n}(m) \ge m \log(2^n/m)$  with equality for *m* of the form  $m = 2^a$ ,  $a = 1, \ldots, 2^{n-1}$ . This is exactly what is provided by Theorem 2.9. As another example consider the Petersen graph *P* (cf. Fig. 4a). It is easily shown that the set of vertices numbered with  $1, \ldots, 5$  is isoperimetric with respect to  $i_2(P)$ . Thus,  $i_2(P) = 2$  and  $\theta_{P^n}(10^n/2) = 10^n/2$ , which matches with the results based on Theorems 2.7 and 2.12 (see [111] for isoperimetric sets with respect to  $i_3$ ).

Concerning random r-regular graphs, Bollobás showed in [29] that the isoperimetric number  $i_1$  of such graphs converges to r/2 as  $r \to \infty$ . More exactly denote

 $i_1(r) = \sup\{\gamma \mid i_1(G) > \gamma \text{ for infinitely many } r\text{-regular graphs } G\}.$ 

It is known (see [29] and [7] for the lower and the upper bound respectively) that

$$\frac{r}{2} - \sqrt{\ln 2} \cdot \sqrt{r} \le i_1(r) \le \frac{r}{2} - \frac{3}{8\sqrt{2}} \cdot \sqrt{r},$$

where the upper bound is valid for sufficiently large n (surely for  $n \ge 40r^9$ ).

Isoperimetric numbers found applications not only for isoperimetric inequalities but also for many other problems. Among such examples is the relation between  $i_2(G)$  and the edge forwarding index shown in [106] and application of  $i_1(G)$  for estimating the crossing number [101]:

$$cr(G) = \Omega((i_1^2(G)p^2/\Delta_G) - p),$$

for a graph G with p vertices. Further results on the isoperimetric numbers of graphs and their applications can be found in [37, 90].

# 3 Generalizations

Here we consider the problem of maximization of I for graphs represented as cartesian products of other (simple) graphs, for which this problem has the nested solutions property. The question is what does it bring for the existence of a nested structure on the whole graph. In the next subsection, we introduce and study an equivalence relation providing a solution of the problem for each graph from its equivalence class if a solution for at least one representative of this class is known. We also show relations between the isoperimetric problems in graphs and minimization of shadows in posets. In the second subsection, we consider maximization of more general functions on graphs and study a phenomenon providing for a number of orders their optimality with respect to maximization of I on cartesian products of  $n \geq 3$  graphs if they are optimal in the case n = 2.

## **3.1** Equivalence relations for graphs and posets

Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be some graphs with  $|V_1| = |V_2| = p$ . We say that these graphs are *I*-equivalent if the problem of maximization of *I* on each of them has a nested structure of solutions and  $I_{G_1}(m) = I_{G_2}(m)$  holds for  $m = 1, \ldots, p$ . Proposition 2.1 shows that any two trees with the same number of vertices are *I*-equivalent.

**Theorem 3.1** (Bezrukov [17]) Let the graphs  $G_i$  and  $H_i$  be *I*-equivalent for i = 1, ..., n. Then

 $I_{G_1 \times \dots \times G_n}(m) = I_{H_1 \times \dots \times H_n}(m)$ 

for each m. Moreover, the graph  $G_1 \times \cdots \times G_n$  has the nested solutions property iff so is for the graph  $H_1 \times \cdots \times H_n$ .

This theorem, in particular, allows to extend the result of Bollobás and Leader (cf. Theorem 2.6) from the cartesian product of chains to the cartesian products of arbitrary trees with the same number of vertices. As an example, consider the trees P and T shown in Fig. 3a and Fig. 3d respectively and take the cartesian products  $P \times P$  and  $T \times T$  shown in Fig. 3b and

Fig. 3e. The optimal order of  $V_P$  is shown in Fig. 3a, which induces a labeling of  $V_{P\times P}$  (see Fig. 3b). The optimal order of  $V_{P\times P}$  is shown in Fig. 3c.

Now consider the optimal order of  $V_T$  in Fig. 3d, which induces a labeling of  $V_{T\times T}$  (Fig. 3e). Taking the vertices of  $T \times T$  in the same order as the corresponding vertices (i.e. vertices with the same labels) of  $P \times P$  (cf. Fig. 3c), one gets an optimal order for  $T \times T$  shown in Fig. 3f.



Figure 3: Optimal orders for trees

The results concerning the edge isoperimetric problem on some special graphs listed in Section 2.1 allow to construct some families of I-equivalent graphs (see [17] for more details).

In [17] some relations between the edge isoperimetric problems on graphs and some extremal problems on posets were studied. Let  $P = (X, \prec)$  be a ranked poset with rank function  $r_P$ . We remind that a subset  $A \subseteq X$  is called *downset* in P if the conditions  $a \in A$  and  $b \prec a$ imply  $b \in A$ . Define weight of A by  $W_P(A) = \sum_{a \in A} r_P(a)$  and consider the problem of finding a downset  $A \subseteq X$  with |A| = m such that  $W_P(A) \ge W_P(B)$  for any downset  $B \subseteq X$  with |B| = m (cf. Section 2.2). Similarly to above denote

 $W_P(m) = \max\{W_P(A) \mid A \subseteq X \text{ is downset with } |A| = m\}.$ 

Now consider a graph  $G = (V_G, E_G)$  on which the problem of maximization of I has a nested structure of solutions. We say that this graph is representable by a ranked poset  $P = (X, \prec)$ 

with  $|X| = |V_G| = p$  if the problem of maximization of W on this poset has a nested structure of solutions and  $I_G(m) = W_P(m)$  holds for m = 1, ..., p. It is shown in [17] that for any graph in question the representing poset does exist. For example the Petersen graph shown in Fig. 4a is representable by the poset shown in Fig. 4b. The labels of vertices represent the corresponding optimal orders.



Figure 4: The Petersen graph (a) and its representing poset (b)

#### Theorem 3.2 (Bezrukov [17])

Let the graph  $G_i$  be representable by a ranked poset  $P_i$ , i = 1, ..., n. Then

$$I_{G_1 \times \dots \times G_n}(m) = W_{P_1 \times \dots \times P_n}(m)$$

for each m. Moreover, a nested structure of solutions for the graph  $G_1 \times \cdots \times G_n$  in the edge-isoperimetric problem exists iff one exists for the poset  $P_1 \times \cdots \times P_n$  in the problem of maximization of W.

Similar relations for the graphs not representable as cartesian products have already appeared in Theorem 2.11 [3, 67], where the problem of maximization of W was solved by continuous methods. In general, the solution of this problem for a ranked poset follows from solution of the shadow minimization problem, if the last one has some nice properties. To formulate this problem we introduce for a poset  $P = (X, \prec)$  and  $P_i = \{x \in X \mid r_P(x) = i\}$  the notion of shadow  $\Delta(A)$  for  $A \subseteq P_i$ :

$$\Delta(A) = \{ x \in P_{i-1} \mid x \prec a \text{ for some } a \in A \}.$$

The problem is to find for fixed i, m a subset  $A \subseteq P_i$  with |A| = m such that  $|\Delta(A)| \leq |\Delta(B)|$  for any  $B \subseteq P_i, |B| = m$ .

Assume that for a poset  $P = (X, \prec)$  there exists a total order  $\mathcal{O}$  of the set X such that for any i, m the subset defined by the initial segment of length m of  $P_i$  in this order has minimal shadow and this shadow itself is an initial segment of  $P_{i-1}$  in the order  $\mathcal{O}$ . Then (cf. [14]) the problem of maximization of W for a poset P has a nested structure of solutions.

Therefore, the known results on the shadow minimization problem allow to get results on the edge isoperimetric problem via the considered representation of graphs by appropriate posets. For further information we refer the reader to the book of Engel [52] containing a survey on shadow minimization problems (see also [53]) and to the survey paper [56].

Under this approach the edge isoperimetric problem for grids and Hamming graphs are reduced to the shadow minimization problem for the star posets [14, 78, 79] and for the lattice of multisets [47] respectively. In the first case, for example, the chain with p vertices is representable by the star poset consisting of one element **0** of rank 0 and p - 1 elements of rank 1, each of them is greater than **0**. It should be mentioned that in the literature (cf. [14, 78, 79]) the dual of the star poset was studied, i.e. the poset obtained from the star poset by inverting the partial order. However, the shadow minimization problem for a poset is equivalent in a sense to such problem on its dual (cf. [14]) and so the order  $\mathcal{I}$  and the order from [14, 78, 79] should be complementary, which is not so easy to see at once. Further applications of the shadow minimization problems to the edge isoperimetric problems can be found in [17, 22].

## **3.2** Maximization of supermodular functions

The ideas of compression and stabilization can be well applied not only for the maximization of the function I on a graph G, but also for many other functions  $\varphi : 2^{V_G} \mapsto \mathbf{R}$ . An example is the wide-known folklore result on finding a subset of vertices  $A \subseteq V_{Q^n}$  with maximal size of  $I^d(A)$ , where  $I^d(A)$  is the set of all d-dimensional subcubes of  $Q^n$  induced by the vertex set A. With the technique of section 2.1 one can prove that the sets extremal with respect to I are also extremal with respect to  $I^d$  for any  $d = 1, \ldots, n$ .

Let us look on this problem from another side. Consider first the weighted poset  $B^n = (V_{Q^n}, \prec, w)$  with the coordinatewise partial relation  $\prec$  and some rank-symmetric nonnegative weight function w, i.e., such that the condition r(a) = r(b) implies w(a) = w(b) for any  $a, b \in V_{Q^n}$ . Thus, a rank-symmetric weight function is determined by a sequence  $w_0, w_1, \ldots, w_n$ , where  $w_i$  is the weight of any element of  $B^n$  of rank i. Consider a problem of finding a downset of  $B^n$  of fixed size and with maximal weight denoted by  $W_w(A)$  and defined as the sum of weights of its elements.

#### Theorem 3.3 (Bernstein, Hopkroft, Steiglitz [13])

Let w be a rank-symmetric weight function on  $B^n$  such that  $w_i \leq w_j$  whenever i < j. Then for any  $m = 1, \ldots, 2^n$  the collection of the first m vertices of  $B^n$  taken in the lexicographic order has maximal weight among all downsets of  $B^n$  with the same cardinality.

It is important that an extremal downset remains the same, regardless of the concrete nondecreasing sequence  $\{w_i\}$ . Now turn back to the problem of maximization of  $I^d$ . It can be checked that the solution can be seen in the class of downsets of  $B^n$ . Let us consider the weight function w defined by  $w_i = \binom{i}{d}$ ,  $i = 1, \ldots, d$ , where we put  $\binom{i}{d} = 0$  if i < d. For this weight function and a downset A one has:  $|I^d(A)| = W_w(A)$ . Therefore, by Theorem 3.3 the extremal sets are the same for any d = 1, ..., n. Clearly, Theorem 3.3 applied for d = 1 implies Theorem 2.2.

Theorem 3.3 was generalized for unimodal weight functions in [6] and [27] contains its generalization for such functions for the lattice of multisets. An interesting (and still unsolved) problem was proposed in [75]: to find a subset of  $Q^n$  containing maximal number of Hamming triangles with sides 1,1,2.

All the proofs concerning maximization of concrete functions we considered up to now have something in common. They are based on the compression and are done by induction on the dimension. The case of dimension 2 lying in the basis of induction claims a special consideration. For maximization of which else functions on cartesian products can we apply this technique ? Ahlswede and Cai noticed in [4] that for the compression it is essential that the function to be maximized is supermodular. Let G be a graph. We call a function  $\varphi : 2^{V_G} \mapsto \mathbf{R}$  supermodular if

$$\varphi(A) + \varphi(B) \le \varphi(A \cup B) + \varphi(A \cap B)$$
 for all  $A, B \subseteq V_G$ ,

and assume that  $\varphi(\emptyset) = 0$ . Clearly, the function  $I^d$  is supermodular for any d.

Let  $G_1, G_2$  be graphs and  $\varphi_i : 2^{V_{G_i}} \mapsto \mathbf{R}, i = 1, 2$  be supermodular functions. For  $A \subseteq V_{G_1 \times G_2}$ we define the function  $\varphi_1 * \varphi_2 : 2^{V_{G_1 \times G_2}} \mapsto \mathbf{R}$  as

$$\varphi_1 * \varphi_2(A) = \sum_{a \in V_{G_2}} \varphi_1(A_1(a)) + \sum_{b \in V_{G_1}} \varphi_2(A_2(b)),$$

where for all  $a \in V_{G_2}$  and for all  $b \in V_{G_1}$ 

$$A_1(a) = \{ c \in V_{G_1} \mid (c, a) \in A \}$$
 and  $A_2(b) = \{ c \in V_{G_2} \mid (b, c) \in A \}.$ 

Since the operation \* is associative, we define the  $n^{th}$  power of  $\varphi$  by  $\varphi^n = \varphi * \cdots * \varphi$ .

Let the problem of maximization of a supermodular function  $\varphi$  on G have a nested structure of solutions. This structure induces a labeling of vertices of G by  $0, 1, \ldots, |V_G| - 1$ , such that for each m and the set of vertices labeled by  $[m] = \{0, \ldots, m-1\}$  holds:  $\varphi(A) \leq \varphi([m])$  for any  $A \subseteq V_G, |A| = m$ . This labeling in turn induces a labeling of the vertex set of  $G^n = G \times \cdots \times G$ . Let us introduce the coordinatewise partial order on the set  $V_{G^n}$  and the downsets with respect to this order. Using compression technique one can prove the following.

#### **Proposition 3.1** (Ahlswede, Cai [4])

For any  $n \ge 1$  and any  $A \subseteq V_{G^n}$  there exists a downset  $B \subseteq V_{G^n}$  with |A| = |B| such that  $\varphi^n(A) \le \varphi^n(B)$ .

So far we have understood when the compression works out. It is a great step forward since the downsets essentially reduce the number of subsets suspicious for optimality (cf. [38]). Nevertheless the question remains what to do further with a downset. If a problem of minimization of  $\varphi^n$  has a nested structure of solutions, we need a definition of a total order to prove that it is the optimal one. However, in some cases one can answer relatively easy whether a given order is optimal. The first step in this direction was done in [4] with respect to the lexicographic order.

**Theorem 3.4** (Ahlswede, Cai [4]) If  $|V_G| \ge 3$ , then for any  $n \ge 2$  the lexicographic order is optimal for  $\varphi^n$  iff it is optimal for  $\varphi^2$ .

Thus, for a given graph G and the lexicographic order we have to check a finite number of cases for  $\varphi^2$  to ensure its optimality for any  $n \ge 2$ . This theorem the authors called the *local-global* principle and it was used in [4] to prove Theorem 2.4.

Consider, for example, a  $3 \times 3$  grid and the problem of minimization of  $\theta$ . It is easily shown that the lexicographic order provides nestedness in this case. Obviously,  $-\theta$  is a supermodular function and maximization of  $-\theta$  is equivalent to minimization of  $\theta$ . Therefore, by Theorem 3.4 the lexicographic order works for minimization of  $\theta$  for  $3 \times \cdots \times 3$  *n*-dimensional grids, although for general grids the problem does not have nested solutions! Similar observation is also valid for  $2 \times \cdots \times 2$  grids (i.e. for the hypercube) and for  $4 \times \cdots \times 4$  grids. However, for  $5 \times 5$  and larger grids no nested solutions exist.

For which other orders does the local-global principle hold ? Let  $|V_G| = p$  and denote  $\delta_{\varphi}(m) = \varphi([m]) - \varphi([m-1]), m = 0, \dots, p-1$ , with  $[-1] = \emptyset$ . First let us study the order  $\mathcal{I}$  defined in Section 2.1.

**Theorem 3.5** (Ahlswede, Bezrukov [2]) Let the order  $\mathcal{I}$  be optimal for  $\varphi^2$  and  $p \geq 3$ . Then  $\mathcal{I}$  is optimal for  $\varphi^n$  for any  $n \geq 3$  iff  $\delta_{\varphi}(0) \leq \delta_{\varphi}(1) = \delta_{\varphi}(2) = \cdots = \delta_{\varphi}(p-1).$ 

Proof.

Due to Proposition 3.1 we consider downsets only. Note that for a downset A one has (see Lemma 3 in [4])

$$\varphi^n(A) = \sum_{(x_1,\dots,x_n)\in A} \sum_{i=1}^n \delta_{\varphi}(x_i).$$
(8)

First, we show that if the order  $\mathcal{I}$  is optimal for  $\varphi^2$ , then

$$\delta_{\varphi}(1) \ge \delta_{\varphi}(2) \dots \ge \delta_{\varphi}(p-1). \tag{9}$$

Indeed, for  $1 \le t consider the sets$ 

$$A = \{0, 1, \dots, t-1\} \times \{0, 1, \dots, t\},\$$
  

$$B = A \cup \{(t, 0)\},\$$
  

$$C = A \cup \{(0, t+1)\}.$$

The sets A, B and C are downsets, |B| = |C| and B is an initial segment of order  $\mathcal{I}$ . Applying (8), the inequality  $\varphi^2(B) \ge \varphi^2(C)$  is equivalent to  $\delta_{\varphi}(t) \ge \delta_{\varphi}(t+1)$ .

Now assume that the order  $\mathcal{I}$  is optimal for  $\varphi^n$  for some  $n \geq 3$  and

$$\delta_{\varphi}(1) = \cdots = \delta_{\varphi}(t-1) > \delta_{\varphi}(t)$$
 for some  $t, 1 < t \le p-1$ .

Denote by A the collection of n-dimensional vectors, which are not greater than the vector  $(t, 1, \ldots, 1)$  in order  $\mathcal{I}$ . Let

$$B = A \setminus \{(t, 1, \dots, 1)\},\$$
  
$$C = A \setminus \{(t - 1, t, \dots, t)\}.$$

Again the sets A, B and C are downsets, |B| = |C| and B is an initial segment of order  $\mathcal{I}$ . Applying (8), the inequality  $\varphi^n(B) \ge \varphi^n(C)$  is equivalent to

$$(n-2)\delta_{\varphi}(t) + \delta_{\varphi}(t-1) \ge (n-1)\delta_{\varphi}(1).$$

From this and (9) follows  $\delta_{\varphi}(t) \geq \delta_{\varphi}(1)$ . A contradiction. Therefore,  $\delta_{\varphi}(1) = \cdots = \delta_{\varphi}(p-1)$ . Furthermore, since the order  $\mathcal{I}$  is optimal for  $\varphi^2$  then

$$\varphi^2(\{(0,0), (0,1), (1,0), (1,1)\}) \ge \varphi^2(\{(0,0), (0,1), (1,0), (2,0)\}).$$

Therefore,

$$2\delta_{\varphi}(1) \ge \delta_{\varphi}(0) + \delta_{\varphi}(2) = \delta_{\varphi}(0) + \delta_{\varphi}(1),$$

from where  $\delta_{\varphi}(0) \leq \delta_{\varphi}(1)$  follows.

On the other hand, if  $\delta_{\varphi}(0) = \delta_{\varphi}(1) = \delta_{\varphi}(p-1)$ , then (9) implies that the function  $\varphi^n$  depends on the size of the downset only, thus theorem is true. Otherwise, if  $\delta_{\varphi}(0) < \delta_{\varphi}(1) = \delta_{\varphi}(p-1)$ , then the proof of optimality of the order  $\mathcal{I}$  for  $\varphi^n$  can be done quite similar to the case  $\delta_{\varphi}(0) = 0$ and  $\delta_{\varphi}(1) = \cdots = \delta_{\varphi}(p-1) = 1$  (cf. the proof of Theorem 2.6 in [1] or [32]).

Note that in the binary case p = 2 the order  $\mathcal{I}$  is the lexicographic order. Counterexamples show that in this case it is not true, that if  $\mathcal{I}$  is optimal for  $\varphi^2$ , then it is optimal for  $\varphi^n$  for any  $n \geq 3$ .

Now let us switch to the simplex order  $\mathcal{H}$  which provides a solution to the vertex isoperimetric problem (cf. [14, 38]). For a vector  $\mathbf{x} = (x_1, \ldots, x_n) \in [p] \times \cdots \times [p]$  denote  $\|\mathbf{x}\| = x_1 + \cdots + x_n$ . We say  $\mathbf{x} >_{\mathcal{H}} \mathbf{y}$  iff

a.  $\|\mathbf{x}\| > \|\mathbf{y}\|$ , or b.  $\|\mathbf{x}\| = \|\mathbf{y}\|$  and  $\mathbf{x} <_{\mathcal{L}} \mathbf{y}$ , where  $\mathcal{L}$  is the lexicographic order.

**Theorem 3.6** (Ahlswede, Bezrukov [2]) Let the order  $\mathcal{H}$  be optimal for  $\varphi^2$  and  $p \ge 2$ . Then  $\mathcal{H}$  is optimal for  $\varphi^n$  for any  $n \ge 3$ .

#### Proof.

Again due to Proposition 3.1 we consider the downsets only. First we show that if  $\varphi$  is optimal for  $\varphi^2$ , then

$$\delta_{\varphi}(0) \ge \delta_{\varphi}(1) \ge \delta_{\varphi}(2) \dots \ge \delta_{\varphi}(p-1), \tag{10}$$

$$\delta_{\varphi}(a) + \delta_{\varphi}(b) = \delta_{\varphi}(c) + \delta_{\varphi}(d) \quad \text{for} \quad a+b = c+d.$$
(11)

Indeed, consider the ball  $B_i^n$  of radius *i* centered in the origin

$$B_i^n = \{(x_1, \dots, x_n) \in [p] \times \dots \times [p] \mid x_1 + \dots + x_n \le i\}.$$

Assuming  $1 \le i denote$ 

$$Q_i = (B_i^2 \setminus \{(0,i)\}) \cup \{(i+1,0)\}$$

and note that  $Q_i$  is a downset. Since  $\mathcal{H}$  is optimal for  $\varphi^2$ , then  $\varphi^2(B_i^2) \ge \varphi^2(Q_i)$ , which with (8) implies  $\delta_{\varphi}(i) \ge \delta_{\varphi}(i+1)$  for  $i \ge 1$ . In order to prove  $\delta_{\varphi}(0) \ge \delta_{\varphi}(1)$  consider instead of  $Q_i$  the set

$$Q_{p-1} = (B_{p-1}^2 \setminus \{(0, p-1)\}) \cup \{(p-1, 1)\}.$$

Then  $Q_{p-1}$  is a downset and  $\varphi(B_{p-1}^2) \ge \varphi(Q_{p-1})$  completes the proof of (10).

In order to prove (11), denote for  $(x, y) \in [p] \times [p]$ 

$$[x,y]_{\mathcal{H}} = \{(a,b) \in [p] \times [p] \mid (a,b) \leq_{\mathcal{H}} (x,y)\},\$$

i.e. an initial segment of the order  $\mathcal{H}$ . For  $i \geq 1$  and  $i > j \geq 0$  consider the set

$$T_i^j = [j, i-j]_{\mathcal{H}} \setminus \{(j+1, i-j-1)\}.$$

Since  $\mathcal{H}$  is optimal for  $\varphi^2$ , then  $\varphi^2(T_i^j) \leq \varphi^2([j+1, i-j-1]_{\mathcal{H}})$ , which with (8) implies

$$\delta_{\varphi}(j) + \delta_{\varphi}(i-j) \le \delta_{\varphi}(j+1) + \delta_{\varphi}(i-j-1).$$
(12)

Applying (12) for  $j = 0, \ldots, i - 1$  one has

$$\delta_{\varphi}(0) + \delta_{\varphi}(i) \le \delta_{\varphi}(1) + \delta_{\varphi}(i-1) \le \delta_{\varphi}(2) + \delta_{\varphi}(i-2) \le \dots \le \delta_{\varphi}(i) + \delta_{\varphi}(0).$$

Thus, the above mentioned equalities are valid, which implies (11).

Now we extend (11) for the case of n > 2 summands, showing that for  $(x_1, \ldots, x_n) \in [p] \times \cdots \times [p]$ the magnitude  $\delta_{\varphi}(\mathbf{x}) = \delta_{\varphi}(x_1) + \cdots + \delta_{\varphi}(x_n)$  is a function of  $i = ||\mathbf{x}||$ , i.e. that

$$\delta_{\varphi}(\mathbf{x}) = \delta_{\varphi}(\mathbf{y}), \quad \text{if } \|\mathbf{x}\| = \|\mathbf{y}\|.$$
(13)

To show this, represent i in the form i = (p-1)s + r with  $0 \le r < p-1$  and consider the vector

$$\mathbf{l}_i = (0, \dots, 0, r, \underbrace{p-1, \dots, p-1}_{s}).$$

We will show that  $\delta_{\varphi}(\mathbf{x}) = \delta_{\varphi}(\mathbf{l}_i)$ . For this assume that  $\mathbf{x} \neq \mathbf{l}_i$ , so there exist some i, j such that  $x_1 = \cdots = x_{i-1} = 0$  with  $x_i > 0$  and  $x_{j+1} = \cdots = x_n = p - 1$  with  $x_j . Consider the vector$ 

$$\mathbf{z} = (x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_n).$$

Applying (11), one has  $\delta_{\varphi}(\mathbf{x}) = \delta_{\varphi}(\mathbf{z})$ , and clearly after a finite number of such replacements one has  $\mathbf{x} = \mathbf{l}_i$  and  $\mathbf{y} = \mathbf{l}_i$  and (13) follows.

Finally, let us show that

$$\delta_{\varphi}(\mathbf{x}) \le \delta_{\varphi}(\mathbf{y}), \quad \text{if } \|\mathbf{x}\| \ge \|\mathbf{y}\|.$$
(14)

Indeed, (14) follows from (13) in the case  $\|\mathbf{x}\| = \|\mathbf{y}\|$ . Otherwise, (14) is a consequence of (13) and  $\delta_{\varphi}(\mathbf{l}_i) \leq \delta_{\varphi}(\mathbf{l}_{i+1})$ , where the last inequality is implied by (10).

Now we are ready to prove the theorem. Assume that a downset  $A \subseteq [p] \times \cdots \times [p]$  is not an initial segment of order  $\mathcal{H}$ . Denote by  $\mathbf{x}$  the largest vector of A in order  $\mathcal{H}$  and by  $\mathbf{y}$  the smallest vector in this order which is not in A. Clearly,  $\mathbf{x} >_{\mathcal{H}} \mathbf{y}$ , and  $\|\mathbf{x}\| \geq \|\mathbf{y}\|$ .

Consider the downset  $B = (A \setminus \{\mathbf{x}\}) \cup \{\mathbf{y}\}$ . Applying (8), the inequality  $\varphi^n(A) \leq \varphi^n(B)$  is equivalent to  $\delta_{\varphi}(\mathbf{x}) \leq \delta_{\varphi}(\mathbf{y})$ , which is true due to (14). After a finite number of such replacements one can transform A into an initial segment of order  $\mathcal{H}$  without decreasing of  $\varphi^n$ , and the theorem follows.

Counterexamples show that in the binary case p = 2 Theorem 3.6 is not true in general.

Let us refer again to the edge isoperimetric problem in the form of maximization on the function I on cartesian products of a connected graph G. One has  $\delta_I(0) = 0$  and  $\delta_I(1) = 1$ . By Theorem 3.5, if the order  $\mathcal{I}$  is optimal for I on  $G^n$ , then  $\delta_I(1) = \cdots = \delta_I(|V_G| - 1)$ . It is easily shown that if the function I satisfies this property, then G contains no cycles and is connected, i.e. G is a tree. Thus, using Theorem 3.2, the order  $\mathcal{I}$  only works for cartesian products of trees (and for no other graphs !). Since  $\delta_I(m) \geq 0$  for all m, there are no graphs at all for cartesian products of which the oder  $\mathcal{H}$  would be optimal with respect to the function I.

# 4 Applications

Each of the three topics considered below is quite broad and requires separate consideration. We do not give a complete survey on these topics, just showing how the isoperimetric methods are applicable.

## 4.1 The wirelength problem

One of the first needs of edge isoperimetric problems was discovered by Harper in [63]. Suppose we have to send the numbers  $0, 1, \ldots, 2^n - 1$  through a binary channel and we have to assign the numbers to vertices of the *n*-cube  $Q^n$ . For example, we may assume that these numbers were taken from the output of an analogue to code digital converter. It is assumed that only single errors are likely in a transmitted word and all *n* positions may be disturbed with probability *p*. If the *n*-tuple assigned to *i* was transmitted and the *n*-tuple assigned to *j* was received, then |i - j| is the absolute value of the error. The goal is to find an assignment so that the average absolute error in transmission is minimized under the condition that the choice of the  $2^n$  numbers is equally probable. Thus one comes to the problem of constructing a bijective mapping  $\varphi: V_{Q^n} \mapsto \{0, \ldots, 2^n - 1\}$  so that the sum  $\sum_{(u,v) \in E_{Q^n}} |\varphi(u) - \varphi(v)|$  is minimized.

Such type problems can be formulated for an arbitrary connected graph G and the sum above

may be referred to the total wirelength in a linear layout of the graph G. The usefulness of the edge isoperimetric problem for the wirelength problem follows from the key identity proved in [63]. Let  $S_{\varphi}(m)$  denote the set of vertices of G labeled by  $\varphi$  with  $0, \ldots, m-1$ . Then

$$\min_{\varphi} \sum_{(u,v)\in E_G} |\varphi(u) - \varphi(v)| = \min_{\varphi} \sum_{m=0}^{2^n} \theta_G(S_{\varphi}(m)) \ge \sum_{m=0}^{2^n} \theta_G(m).$$
(15)

Therefore, if the problem of minimization of  $\theta_G$  has the nested structure of solutions, then the corresponding ordering provides equality in (15) and so a solution for the wirelength problem.

This approach was used in [63] to show that in the wirelength problem for  $Q^n$  the lexicographic order provides a (essentially unique) solution and that the wirelength equals  $2^{n-1}(2^n - 1)$ . In [13] it was shown that the above coding works for any mapping  $\varphi : V_{Q^n} \mapsto \{a_0, \ldots, a_{2^n-1}\}$ with  $0 \leq a_0 \leq \cdots \leq a_{2^n-1}$  if the probability p is small enough. Here, the number  $a_i$  should be assigned with the vertex corresponding to the binary expansion of i. In [15] the case was considered where some t, 0 < t < n, positions in all codewords are error-free. If these t positions are the first ones, then it can be shown (cf. [15]) that the lexicographic order works as well and is essentially unique (up to isomorphism).

The result concerning the wirelength of the *n*-cube was extended in [97] for the Hamming graph  $H^n(a, \ldots, a)$ , where it is shown that the wirelength equals  $(a + 1)a^n(a^n - 1)/6$ . Another extension concerns minimization of  $\sigma_2(G) = \sum_{(u,v)} (\varphi(u) - \varphi(v))^2$ , where the sum runs over all edges of G. For  $Q^n$  (see [48]), it is shown that the lexicographic order provides a solution. The proof is, however, much more complicated and is based on usage of Fourier transforms. For general graphs with p vertices it is known [72] that  $\sigma_2(G) \ge \lambda_G(p^2 - 1)p/6$ .

The ideas of compression and stabilization we presented here were started to be studied systematically by Harper in [66]. This led to a nice theory which he calls stabilization theory and will be further developed in his forthcoming book [38] (it should be mentioned that he used the term "stabilization" in a different sense). Shortly, Harper introduced a set of geometric transformations in Euclidean space based on reflections, which do not increase the edge lengths  $|\varphi(u) - \varphi(v)|$  and thus not the wirelength either. Application of these transformations with respect to the edge isoperimetric problem or to the wirelength problem leads to a solution represented by stable configurations, which significantly reduce the number of mapping to be considered. With the help of his theory Harper solved the wirelength problems for the dodecahedron, icosahedron and the 24-cell [66], for the 600-cell [11] and for the binary de-Bruijn graph of dimension 4 [65]. In the last case the lower bound based on identity (15) is 74, and the absence of a nested structure of solutions of the edge isoperimetric problem for this graph led to increase this bound up to 76 (the wirelength for this graph has to be even), which is provided by a corresponding numbering. However, for the higher-dimensional case new ideas are required. By using (15) and Theorem 2.7 it is shown in [24] that the wirelength of the  $n^{th}$  power of the Petersen graph equals  $\frac{37}{82} \cdot 100^n + \frac{72}{82} \cdot 18^{n-1} - \frac{1}{2} \cdot 10^n$ .

The wirelength problem for the 2-dimensional  $n \times m$  grid  $(n \leq m)$  [93] has an interesting solution (see also [88] for square grids and [55]). Although the edge isoperimetric problem does not have the nested solutions property, it was possible (using similar approach based on compressions) to find the exact value of the wirelength. The optimal numbering is schematically shown in Fig. 5. The numbering starts with the left lower corner of the grid and consequently fills the areas  $A_1, A_2, \ldots, A_7$  (cf. Fig. 5a), where  $A_1, A_3$  are  $a \times a$  squares and  $A_5, A_7$  are  $a' \times a'$  squares (a and a' will be specified below).

The numbering of the areas is shown in Fig. 5b. First we number a square after a square filling a row after a column until we fill the square  $A_1$  with the side length a. Then we proceed with the area  $A_2$  numbering it consecutive rows from bottom to top and from left to right. After that we number the  $a \times a$  square  $A_3$  with the reversed order with respect to  $A_1$ . Next we number the columns of  $A_4$  from bottom to top and from left to right. The numbering is completed by numbering of the areas  $A_5 - A_7$  in the similar way.



Figure 5: The wirelength numbering of a grid

It is shown in [93] that a and a' can be chosen arbitrarily from the set

$$a, a' \in \left\{ \left[ n - \frac{1}{2} - \sqrt{\frac{n^2}{2} - \frac{n}{2} + \frac{1}{4}} \right], \left[ n + \frac{1}{2} - \sqrt{\frac{n^2}{2} - \frac{n}{2} + \frac{1}{4}} \right] \right\}.$$

If (square root + 1/2) in this formula is an integer, then *a* and *a'* can differ in 1 and it makes no difference how to choose them. Otherwise they are defined uniquely. Moreover, the wirelength of the grid equals

$$-\frac{2}{3}a^{3} + 2na^{2} - \left(n^{2} + n - \frac{2}{3}\right)a + m(n^{2} + n - 1) - n$$

This result implies a formula for the wirelength of the  $n \times m$  torus T(n,m) (i.e. the cartesian product of two cycles with n and m vertices), since as it is easily shown, the wirelength of the torus is twice larger the one of the  $n \times m$  grid (see [64] for some results concerning continuous approximation of the torus and its wirelength).

It is known how to solve the wirelength problem for the cartesian product of a cycle with a chain (a 2-dimensional cylinder) [95] and for the complete *p*-partite graph [94]. In the last case a nested structure of solutions in the edge isoperimetric problem (cf. Theorem 2.1) provides a solution due to (15). In [72, 73] it is shown that the wirelength of a graph G with p vertices is at least  $\lambda_G(p^2 - 1)/6$ .

For further information concerning the wirelength problem for other graphs we refer to the surveys [40, 41, 72, 73, 96]. In [38] one can find formulation and some results on the wirelength problem for ranked posets and in [71] the solution of this problem for the Boolean lattice.

## 4.2 The bisection width and edge congestion

Edge isoperimetric problems often and naturally arise in various problems on networks. For estimating the communication complexity or the layout (cf [80]), for example, it is important to know what is the least number of edges one has to cut in order to split a given graph into 2 parts with equal number of vertices. This parameter is known as the bisection width of G(denotation bw(G)).

As another example let G and H be graphs and consider all injective mappings  $\varphi : V_G \mapsto V_H$ , which we call embeddings of G into H. We assume that an embedding is equipped with a routing scheme R, which maps edges of G into paths of H. For an edge  $e \in E_H$  denote by  $econ_{\varphi,R}(e)$  the number of paths in the routing scheme R passing through the edge e and let

$$econ(G, H) = \min_{\varphi, R} \max_{e \in E_H} econ_{\varphi, R}(e)$$

This parameter is called edge congestion and is well studied if H is a path with  $|V_G|$  vertices. In this case it is simply called cutwidth of G (denotation cw(G)). One has

$$bw(G) = \theta_G(\lfloor |V_G|/2 \rfloor),$$
  

$$cw(G) \ge \max_m \theta_G(m).$$
(16)

The isoperimetric sets providing the values of the bisection width and cutwidth for the *n*-cube and the Hamming graph follow from Theorems 2.2 and 2.3 respectively. However, computation of these values makes some difficulties (see [10, 62, 97]):

$$bw(G^{n}(p,...,p)) = \begin{cases} p^{n-1}, & p \text{ even} \\ (p^{n}-1)/(p-1), & p \text{ odd.} \end{cases}$$
$$bw(H^{n}(p,...,p)) = \begin{cases} p^{n+1}/4, & p \text{ even} \\ (p+1)(p^{n}-1)/4, & p \text{ odd.} \end{cases}$$

$$(p^{n}-1)/(p-1) \ge cw(G^{n}(p,\ldots,p)) \ge \begin{cases} (p+2)(p^{n}-1)/(p^{2}+p), & p \text{ even, } n \text{ even} \\ ((p+2)p^{n-1}-1)/(p+1), & p \text{ even, } n \text{ odd} \\ (p^{n}-1)/(p-1), & p \text{ odd.} \end{cases}$$

$$cw(H^{n}(p,\ldots,p)) = \begin{cases} p(p+2)(p^{n}-1)/(4p+4), & p \text{ even, } n \text{ even} \\ p^{2}((p+2)p^{n-1}-1)/(4p+4), & p \text{ even, } n \text{ odd} \\ (p+1)(p^{n}-1)/4, & p \text{ odd.} \end{cases}$$

Let us also mention the results of [24] and [102] concerning the powers  $P^n$  of the Petersen graph and the 2-dimensional torus T(n, m) respectively, where it is shown that the lower bound (16) is strict is these cases and

$$cw(P^{n}) = \begin{cases} (6.25) \cdot 10^{n-1} + (2^{n-1} - 4)/12, & n \text{ odd} \\ (6.25) \cdot 10^{n-1} + (2^{n-1} - 8)/12, & n \text{ even}, \end{cases}$$
$$cw(T(n,m)) = \min\{2n+2, 2m+2\}.$$
(17)

For r-regular graphs with  $|V_G| = p$  an alternative lower bound is proposed in [19]

$$bw(G) \ge p \cdot \frac{r(r+\lambda_G-1) - r\sqrt{(r-1)(2\lambda_G+r-1)}}{2\lambda_G}.$$
(18)

This bound is better than the lower bound provided by Theorem 2.12 if  $\lambda_G$  is small. In particular, for a family of *r*-regular graphs  $\{G_p\}$  with r = r(p) and  $\lambda_{G_p}/r \to 0$  as  $p \to \infty$  the right hand side of (18) asymptotically equals  $\frac{p}{4} \cdot \frac{r}{r-1}$ .

In general, some other approaches for estimating the bisection width and cutwidth of graphs are known (see e.g. [80] and [107] for application of spectral methods). Further information on this topic can be found in the surveys [40, 41].

If the graph H is not a path, just a few exact results on estimating the edge congestion are known. A general lower bound is due to the isoperimetric approach [20]:

$$econ(G, H) \ge \max_{m} \frac{\theta_G(m)}{\theta_H(m)}.$$
 (19)

Let C(k) denote a cycle with k vertices and consider an embedding of a graph G into  $C(|V_G|)$ . We introduce the cyclic cutwidth and cyclic wirelength of G as

$$ccw(G) = econ(G, C(|V_G|)),$$
  

$$cwl(G) = \min_{\varphi, R} \sum_{e \in C(|V_G|)} econ_{\varphi, R}(e)$$

Now let  $T(m,n) = C(m) \times C(n)$  be a two-dimensional torus with  $m \ge n$ . For embedding of T(m,n) into C(mn) the lower bound (19) (with usage of (17)) is not strict, since

$$ccw(T(m, n)) = \min\{m + 2, n + 2\}$$

as it is shown in [102] with nice techniques based on consideration of all minimal cuts of the torus. A more difficult problem is to compute the cyclic cutwidth of grids. Recently it is proved in [104] that for an  $m \times n$  grid with  $m \ge n \ge 3$ 

$$ccw(P_m \times P_n) = \begin{cases} n-1, & \text{if } m = n \text{ is even,} \\ n, & \text{if } n \text{ is odd or } m = n+2 \text{ is even,} \\ n+1, & \text{otherwise} \end{cases}$$

Thus, the lower bound (19) is not strict for grids too. It would be of interest to compute the cyclic wirelength of grids and tori. Another interesting problem is to embed  $Q^n$  into  $C(2^n)$ . In [10] it is shown that

$$ccw(Q^n) \le (5 \cdot 2^{n-2} - 2 + (n \mod 2))/3.$$

The construction consists of isomorphic embedding four copies of  $Q^{n-2}$  into four segments of the cycle using the wirelength embedding into the line and connecting the corresponding vertices of  $Q^{n-2}$ 's by 4-cycles. It is conjectured that this embedding is optimal and so the bound (19) is not strict. The same construction provides the upper bound in

$$\frac{1}{3} \le \lim_{n \to \infty} \frac{cwl(Q^n)}{4^n} \le \frac{3}{8}$$

The lower bound follows from isoperimetric arguments. Recently it is proved in [59] that the construction above (and, thus, the upper bound) provides an exact answer.

For some graph classes, however, the cutwidth and the cyclic cutwidth are the same. For example, it is the case for trees [39, 82]. In [26] it is shown that also the wirelength and the cyclic wirelength of any tree are equal. More about embeddings into cycle one can find in [58, 70, 81, 82, 112].

Embedding of  $Q^n$  into grids was studied in a number of papers (cf. [41, 113]), where the order of the edge congestion was determined. Its exact value is found in [20]:

$$econ(Q^n, G^d(2^{n_1}, \dots, 2^{n_d})) = \begin{cases} (2^{n_d+1}-1)/3, & n_d \text{ odd} \\ (2^{n_d+1}-2)/3, & n_d \text{ even} \end{cases}$$

where  $n_1 \leq \cdots \leq n_d$  and  $n_1 + \cdots + n_d = n$ . Therefore, in this case the lower bound (19) is strict.

## 4.3 Graph partitioning problems

Let an edge cut partition the vertex set of a graph G into k parts  $A_1, \ldots, A_k$  with

$$\lfloor |V_G|/k \rfloor \le |A_i| \le \lceil |V_G|/k \rceil.$$
<sup>(20)</sup>

Denote  $\nabla(G, k) = \min \left| \bigcup_{i=1}^{k} \theta_G(A_i) \right|$ , where the minimum runs over all partitions of  $V_G$  satisfying (20). Such problems arise, for example, in load balancing under distribution of tasks in multiprocessor computing systems.

The edge isoperimetric problems are naturally applied to the k-partitioning due to the lower bound

$$\nabla(G,k) \ge \frac{k}{2} \min\left\{\theta_G\left(\left\lfloor \frac{|V_G|}{k}\right\rfloor\right), \theta_G\left(\left\lceil \frac{|V_G|}{k}\right\rceil\right)\right\}.$$
(21)

Paper [18] contains some bounds and asymptotic results concerning  $\nabla(Q^n, k)$ . It is shown that in some cases the lower bound (21) is strict and that for  $a > b \ge 0$ :

$$\lim_{d \to \infty} \frac{\nabla(Q^n, 2^a + 2^b)}{2^n} = \frac{a2^{a-1} - b2^{b-1}}{2^a - 2^b},$$
$$\lim_{d \to \infty} \frac{\nabla(Q^n, 2^a - 2^b)}{2^n} = \frac{a2^{a-1} - b2^{b-1} - 2^b}{2^a - 2^b}.$$

It is interesting to notice that the function  $\nabla(Q^n, k)$  is not monotone with k. Theorems 2.3 and 2.4 allow to extend these results for partitioning the Hamming graphs and the graphs  $F^n(p, \ldots, p)$  (cf. Section 2.1). For example, [23]

$$\nabla(H^n(p,\ldots,p), p^a + p^b) \sim \frac{ap^{a-1} - bp^{b-1}}{p^a - p^b} (p-1)p^n$$

as  $a, b, p = const, d \to \infty$  and  $d > a > b \ge 0$  and

$$\nabla(H^n(p,\ldots,p),k) \sim \frac{p^{n+1}}{2} \cdot \frac{k-1}{k},$$
  
$$\nabla(F^n(p,\ldots,p),k) \sim \frac{p^{n+1}}{16} \cdot \frac{k-1}{k},$$

as  $p \to \infty$  and n, k = const.

The lower bound (21) combined with continuous approximation of the grid is used in [25] to show that

$$np^{n-1}\left(\sqrt[n]{k}-1\right) \le \nabla(G^n(p,\ldots,p),k) \le np^{n-1}\left(\sqrt[n]{k}+c_n\right)$$

with some constant  $c_n$  depending on n only. Moreover, some heuristics are proposed in [25] which provide better results for small values of k (see also [86]).

For partition of general connected graphs with p vertices it is proved in [51] that, in particular,

$$\nabla(G,k) \ge \frac{p}{2k} \sum_{i=2}^{k} \lambda_i,\tag{22}$$

where  $0 < \lambda_2 \leq \cdots \leq \lambda_k$  are the eigenvalues of the Laplacian of G. Further application of spectral approach to k-partitioning of weighted graphs and hypergraphs can be found in [28].

Another version of the graph partition problem claims to find a partition  $A_1, \ldots, A_k$  of the vertex set of G with  $2 \le k \le c$  such that  $A_i \ne \emptyset$  and  $\max_i |\theta(A_i)|$  is minimized. We denote this minimal value by B(G, c).

Such a problem arises in the pin limitation problem or in the I/O complexity problem [49]. Constructions of k-partitions considered above can be used for obtaining upper bounds for B(G, k). Concerning the lower bounds, an interesting technique is proposed in [49] involving the time required for sorting or permuting in networks. In particular, [49] contains an inequality for the grid in the form

$$B(G^{n}(p,\ldots,p),k) \ge np^{n-1}(k^{1/n}-1)/k.$$

For further information on results and techniques of k-partitioning of graphs we refer the reader to the papers [49, 100].

# 5 Concluding remarks

We considered edge isoperimetric problems on graphs. In the sections above we presented some known results for concrete graphs, described some standard and new tools and methods for their solution and listed some applications. We concentrated on cartesian products of graphs, where many of deep and general results have been found in recent years. Restricted volume of the paper did not allow to consider many other results and related problems. Here we just mention them briefly and give the references.

A very interesting paper of Shahrokhi and Székely [105] is devoted to a technique for estimating the isoperimetric number  $i_1$  of graphs which is based on concurrent flows. The lower bounds presented there improve some results of Babai and Szegedy [9] on isoperimetric numbers of edge-transitive graphs (see also [103]). The paper [74] of Karisch and Rendl provides a new approach for getting lower bounds for the k-partitioning. This approach is based on a special representation of the size of the cut as the trace of some related matrix. Now minimization of the cut is reduced to minimization of a linear function (the trace) over a set of matrices satisfying some restrictions, which is done by using semidefinite programming. It is also shown how to reformulate into these terms some known eigenvalue bounds (e.g. (22)) and that the new approach gets better lower bounds. In [68] the authors applied the projection technique for deriving spectral lower bounds for the vertex separators and the wirelength. The obtained bounds are complicated enough but provide better results for the wirelength than the spectral bound from Section 4.2.

Concerning related problems, we did not touch a broad area of isoperimetric constants for product Markov chains and probability measures. Many results and references on this subject can be found in [44, 69, 103, 111] and the survey of Talagrand [110]. A kind of an isoperimetric constant for special oriented graphs was studied by Plünnecke (cf. chapter 7 in [98]). He derived some inequalities involving these constants which have powerful consequences in additive number theory.

The isoperimetric approach provides a powerful tool to solution of many discrete extremal problems. Among them is the problem of maximization of the function

$$\min\{\operatorname{dist}_H(f(u), f(v)) \mid (u, v) \in E_G\}$$

over all one-to-one mappings  $f: V_G \mapsto V_H$  [87]. Bollobás and Leader [33] used a combination of this approach and Menger's theorems for constructing edge-disjoint paths connecting the vertices of two sets of  $Q^n$  of the same cardinality. In a very recent paper Sýkora and Vrt'o [108] found a lower bound on the bipartite crossing number of graphs, which involves the function  $\theta$ . (cf. e.g. [92]), where the eigenvalue technique is applied for

Let us mention some research directions:

- 1. Specify the graphs for whose cartesian products the lexicographic order provides a solution of the edge isoperimetric problem;
- 2. For which further orders does the local-global principle hold ?
- 3. We have showed how the shadow minimization problem can be applied to the edge isoperimetric problems. The general question is: how could an edge isoperimetric problem on a graph be used to solve the shadow minimization problem on the representing poset ?

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