On the Bandwidth of 3-dimensional Hamming graphs

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Abstract

This paper presents strategies for improving the known upper and lower bounds for the bandwidth of Hamming graphs $(K_n)^d$ and $[0,1]^d$. In particular, it is shown that the the bandwidth of $K_6 \times K_6 \times K_6$ is exactly 101. The same numbering strategy lowers the upper bound on the bandwidth of the continuous Hamming graph, $[0,1]^3$, from .5 to .4497. A lower bound of .4439 on $bw([0,1]^3)$ follows from known isoperimetric inequalities and a related dynamic program is conjectured to raise that lower bound to $4/9 = .4444...$.

Keywords: Combinatorial optimization

1 Introduction

A *simple graph*, $G = (V, E)$, consists of a set, V, of *vertices*, and a set, $E \subseteq \binom{V}{2}$, of (unordered) pairs of vertices called *edges*. Each edge is *incident to* (contains) two distinct vertices.

Example 1 *K_n, the (standard) complete graph on n vertices, has $V = [n] = \{0, 1, \ldots, n-1\}$ and $E = \binom{[n]}{2}$.*

Example 2 *A product of complete graphs, $K_{n_1} \times K_{n_2} \times \cdots \times K_{n_d}$, is called a Hamming graph since each pair of vertices is connected by an edge iff they are at Hamming distance one from each other (i.e. the two vertices differ in exactly one coordinate).*

A *numbering of G* is a one-to-one and onto function, $\eta : V \mapsto \{1, 2, \ldots, n\}$, where $n = |V|$. The *bandwidth of η* is then

$$bw(\eta) = \max_{(v,w)\in E} |\eta(v) - \eta(w)|,$$

and the *bandwidth of G* is

$$bw(G) = \min_{\eta} bw(\eta).$$

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Example 3 *Every numbering of K_n has the same bandwidth, so $bw(K_n) = n - 1$.*

In general, calculating $bw(G)$ is an intractable problem (NP-hard since the decision problem $bw(G) < k$ is NP-complete (see [4])), but it has been solved for a few families of graphs having special properties. Among these is $K_2^d = K_2 \times K_2 \times \cdots \times K_2$ (aka *the graph of the d-dimensional cube*). In [5] it is shown that

$$bw\left(K_2^d\right) = \sum_{k=0}^{d-1} \binom{k}{\lfloor k/2 \rfloor}.$$

For a survey of results on the bandwidth problem see [3]. The bandwidth of the Hamming graph, K_n^d, $n > 2$, has been an outstanding problem for at least forty years and recently acquired additional interest by being applied to multicasting (see [2] and [1]). In [7] it is shown that for n, d even,

$$\binom{d}{d/2}\left(\frac{n}{2}\right)^d \leq bw\left(K_n^d\right) \leq bw\left(K_2^d\right)\left(\frac{n}{2}\right)^d + \frac{n}{2} - 1.$$

This indicates that for fixed d the order of magnitude of $bw\left(K_n^d\right)$ is $\Theta(n^d)$ as $n \to \infty$ and suggests passing to the continuous limit.

2 The Continuous Limit

In solving the bandwidth problem on K_n^d one may assume that the numbering, η, is monotone increasing as a function of the coordinates, $0 < 1 < \cdots < n-1$. This is a special case of the theory of compression presented in Chapters 3 and 6 of [8]. It is shown there that in many interesting cases of the bandwidth problem, the vertex set, V, may be given a partial order and numberings restricted to be monotone with respect to that partial order (*i.e.* if $x \leq y$ then $\eta(x) \leq \eta(y)$). Passing to a continuous limit, we define a *numbering* of the *continuous Hamming graph,* $[0,1]^d$, to be a monotone, measure-preserving (*i.e.* for all measurable $A \subseteq [0,1]$, $|\eta^{-1}(A)| = |A|$), upper semicontinuous function, $\eta : [0,1]^d \mapsto [0,1]$. As in the finite case, $v, w \in [0,1]^d$ have an edge between them if they differ in exactly one coordinate. Also

$$bw(\eta) = \max_{(v,w)\in E} |\eta(v) - \eta(w)|,$$

and

$$bw\left([0,1]^d\right) = \min_{\eta} bw(\eta).$$

Example 4 *For $d = 1$ the identity, $\iota(x) = x$, is the only monotone measure-preserving function, $\eta : [0,1] \mapsto [0,1]$, so $bw([0,1]) = 1$.*

Theorem 1 $\lim_{n\to\infty} bw\left(K_n^d\right)/n^d \geq bw\left([0,1]^d\right)$.

Proof: Every monotone numbering $\eta : K_n^d \mapsto \{1, \ldots, n^d\}$ may be "blown up" to a numbering $\overline{\eta} : [0,1]^d \mapsto [0,1]$ by filling in each cube of side $1/n$ whose maximum element is $\frac{1}{n}x =$

$(x_1/n, x_2/n, \ldots, x_d/n)$ with values between $(\eta(x)-1)/n^d$ and $\eta(x)/n^d$ in a monotone and measure-preserving way. Then $bw\,(\overline{\eta}) \leq (bw\,(\eta)+1)\,/n^d$ so

$$\lim_{n\to\infty} bw\left(K_n^d\right)/n^d \geq bw\left([0,1]^d\right).$$

□

We believe that equality should hold in Theorem 1. We had thought to prove it by showing that the "blown up" numberings are dense in the set of all numberings, but a colleague (at UCR), Jim Stafney, found a counterexample. If the reverse inequality does hold, then $bw\left([0,1]^d\right)$ is equal to the leading coefficient (of n^d) in the asymptotic series for $bw\left(K_n^d\right)$ as $n\to\infty$.

Theorem 2 *For d even,* $\binom{d}{d/2}/2^d \leq bw\left([0,1]^d\right) \leq \sum_{k=0}^{d-1}\binom{k}{\lfloor k/2\rfloor}/2^d$.

Proof: The upper bound for $bw\left([0,1]^d\right)$ follows from Theorem 1 and the last two displayed formulas in Section 1. The lower bound follows from the argument (in [7]) that $\binom{d}{d/2}\left(\frac{n}{2}\right)^d \leq bw\left(K_n^d\right)$. The left-hand side of this inequality was given by the solution of the vertex-isoperimetric problem on the continuous Hamming graph, so it applies equally well to $bw\left([0,1]^d\right)$. □

Example 5 $bw\left([0,1]^2\right) = 1/2$*, since for $d=2$ the lower and upper bound of Theorem 2 are both $1/2$. One numbering of $[0,1]^2$ with bandwidth $1/2$ is defined as follows:*

$$\eta(x,y) = \begin{cases} y/2 & \text{if } (x,y)\in[0,1/2]\times[0,1] \\ (1+y)/2 & \text{if } (x,y)\in(1/2,1]\times[0,1] \end{cases}$$

That η is measure-preserving follows from the fact that for $t\in[0,1/2]$, $|\eta^{-1}([0,t])| = |\{(x,y) \mid 0\leq x\leq 1/2 \text{ and } 0\leq y\leq 2t\}| = (1/2)2t = t$ and similarly for $t\in[1/2,1]$. In Figure 1 the solid lines represent level curves of η and the dashed line divides the square into two parts where the function takes values less than $1/2$ and greater than $1/2$. The numbering is not symmetric (i.e. invariant under interchange of coordinates), but is self-dual (i.e. invariant under the map that sends (x,y) to $(1-x,1-y)$ and $\eta(x,y)$ to $1-\eta(1-x,1-y)$.

To the reader unfamiliar with measure theory, it may seem strange that that our definition of "numbering" for the continuous Hamming graph does not require η to be one-to-one. However, Theorem 1 and the remarks following it are strong evidence that this definition is natural and useful.

Example 6 *Another optimal numbering, both symmetric and self-dual, is*

$$\upsilon(x,y) = \begin{cases} t^2 & \text{if } (x,y)\in[0,1/2]\times[0,1/2], \\ 1/2-(1-t)^2 & \text{if } (x,y)\in([0,1/2]\times[1/2,1])\cup([1/2,1]\times[0,1/2]) \\ & \quad \text{and } y\leq 1-x, \\ 1/2+u^2 & \text{if } (x,y)\in([1/2,1]\times[0,1/2])\cup([0,1/2]\times[1/2,1]) \\ & \quad \text{and } y>1-x, \\ (1-u)^2 & (x,y)\in[1/2,1]\times[1/2,1]. \end{cases}$$

where $t=\max\{x,y\}$ and $u=\min\{x,y\}$. The level curves of υ are shown in Figure 2.

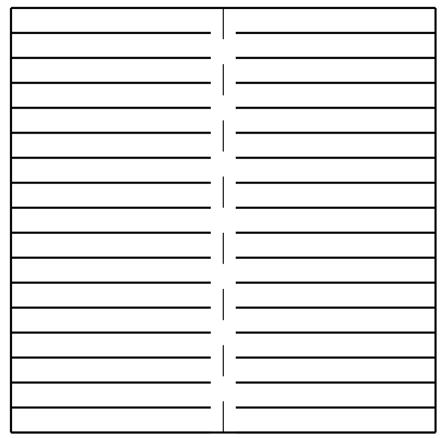

Figure 1: A self-dual asymmetric numbering of $[0,1]^2$ with bandwidth 1/2

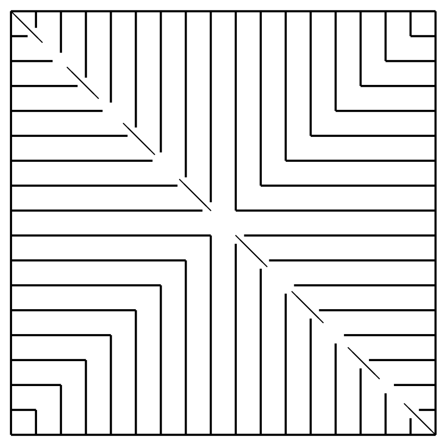

Figure 2: A self-dual and symmetric numbering of $[0,1]^2$ with bandwidth 1/2

Example 7 *As $d \to \infty$, the lower and upper bounds in Theorem 2 are asymptotically equal so*

$$bw\left([0,1]^d\right) \simeq \sqrt{\frac{2}{\pi d}}, \text{ as } d \to \infty.$$

(see [7] for details).

Up to this point in our discussion of the bandwidth problem on the continuous Hamming graph we have followed [7] in assuming that the dimension, d, is even. This was a simplifying assumption made because of the author's conjecture that the upper bound, $bw\left([0,1]^d\right) \leq \sum_{k=0}^{d-1} \binom{k}{\lfloor k/2 \rfloor}/2^d$, is sharp. The same formula for the upper bound holds in odd dimensions and he expected that the lower bound could be improved to meet it. However, the lower bound turns out to be closer to the truth, so we must deal with its logical complexity. The lower bound is actually two bounds, which happen to coincide in even dimensions but not odd.

In [8], Section 4.5.2, it is shown that for any (finite) graph, G,

$$bw(G) \geq \min_{\eta} \max_{0 \leq k \leq n} |\Phi(S_k(\eta))|.$$

where

$$S_k(\eta) = \{v \in V \mid \eta(v) \leq k\}$$

and for $S \subseteq V$,

$$\Phi(S) = \{w \in V - S \mid \exists (v,w) \in E \text{ and } v \in S\}.$$

$\Phi(S)$ is called the *vertex-boundary of* S. The problem of computing

$$\min_{\eta} \max_{0 \leq k \leq n} |\Phi(S_k(\eta))|$$

is a dynamic program, *i.e.* minimum path problem for which Bellman's Principle of Optimality holds (See [8], Chapter 2). It is easy to see that

$$\min_{\eta} \max_{0 \leq k \leq n} |\Phi(S_k(\eta))| \geq \max_{0 \leq k \leq n} \min_{\substack{S \subseteq V \\ |S|=k}} |\Phi(S)|.$$

In graph theory, the problem of calculating $\min_{\substack{S \subseteq V \\ |S|=k}} |\Phi(S)|$ for $0 \leq k \leq n$ is known as the *vertex-isoperimetric problem* (VIP). In general the VIP is just as intractable (*i.e.* NP-complete) as the bandwidth problem, but it (the VIP) has been solved [6] for the continuous Hamming graph, $[0,1]^d$.

When the vertex-set, V, is partially ordered and numberings must be monotone functions, then their initial segments, $S_k(\eta)$, are ideals in the partial order (a subset, S, of a poset $\mathcal{P}$, is called an *ideal* if $x \leq y \in S$ implies $x \in S$). Certain ideals are of particular interest in the continuous Hamming graphs: In $[n]^d$ *the Hamming ball of radius* r *centered at* 0^d*)* is the set

$$HB(n,r,d) = \left\{x \in [n]^d : |\{i : x_i > 0\}| \leq r\right\}.$$

A monotone onto function $\varphi : [n] \to [m]$ is called a *quotient of* n *by* m and extends naturally to $\varphi : [n]^d \to [m]^d$ by defining

$$\varphi(x_1, x_2, \ldots, x_d) = (\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_d)).$$

Then *the quotient Hamming ball*,

$$QHB(n,m,r,d;\varphi) = \varphi^{-1}(HB(m,r,d)).$$

These definitions also makes sense when $n = \infty$, *i.e.* $[n]$ is replaced by $[0,1]$. In [6] it is shown that the quotient Hamming balls $QHB(\infty,2,r,d;\varphi)$, determined by the parameter $t = \max \varphi^{-1}(0)$, are the critical sets for the VIP on $[0,1]^d$. The volume of $QHB(\infty,2,r,d;\varphi)$ is

$$v\,(r,d;t) = \sum_{i=0}^{r} \binom{d}{i} t^{d-i}(1-t)^i$$

and its boundary is

$$|\Phi(r,d;t)| = \binom{d}{r+1} t^{d-r-1}(1-t)^{r+1}.$$

Each quotient Hamming ball minimizes the vertex-boundary, $|\Phi(S)|$, for some interval of values of the volume.

For dimensions, d, that are even, the solution of the VIP on $[0,1]^d$ gives the bound $\binom{d}{d/2}/2^d \leq bw\left([0,1]^d\right)$. Since there is a nested family of sets, the quotient Hamming balls of radius $(d/2)-1$ with $0 \leq t \leq 1$, that achieve this lower bound at $t = 1/2$, the dynamic programming lower bound and the VIP lower bound coincide. For odd dimensions there is no nice formula for the VIP lower bound. It is the common value of the boundary functionals of the quotient Hamming balls of radius $(d-3)/2$ and $(d-1)/2$. The point where their graphs cross may be found by solving equations. For odd d the solution of the dynamic program is not known for certain, but the quotient Hamming balls of radius $(d-3)/2$ (as well as those of radius $(d-1)/2$) are conjectured to be optimal. There is a beautiful formula for their common maximum value,

$$\binom{d}{\lceil (d-1)/2\rceil} \lceil (d-1)/2\rceil^{\lceil (d-1)/2\rceil} \lfloor (d+1)/2\rfloor^{\lfloor (d+1)/2\rfloor},$$

which holds for even d as well as odd (note that $\forall d$, $\binom{d}{\lceil (d-1)/2\rceil} = \binom{d}{\lfloor (d+1)/2\rfloor}$. Some values of these lower (VIP and (conjectured) dynamic programming (DP)) and upper (UB) bounds are given in the following table (UB and DP were calculated from the formulas given above in this section. For odd dimensions VIP was calculated as the crossover value for boundary functions of the quotient Hamming balls of radius $(d-3)/2$ and $(d-1)/2$ (see [6] for details)). The last column give the values of $\underline{\Delta} = \frac{\mathrm{UB}-\mathrm{VIP}}{\mathrm{VIP}} = \frac{\mathrm{UB}}{\mathrm{VIP}} - 1$, the relative difference between the best upper and lower bounds. Note that $\underline{\Delta}$ increases up to a maximum at $d = 5$ and then decreases, going to zero at infinity (this follows from Example 6).

d	VIP	DP	UB	$\underline{\Delta}$
1	1	1	1	0
2	.5	.5	.5	0
3	.4439	.4444	.5	.1263
4	.3750	.3750	.4375	.1667
5	.3454	.3456	.4062	.1760
6	.3125	.3125	.3594	.1501
7	.2937	.2938	.3359	.1437
8	.2734	.2734	.3047	.1145
9	.2602*	.2602*	.2891	.1111
10	.2461	.2461	.2676	.0874

* These two values are not actually equal, but are the same when rounded off to four decimal places.

Table 1: Bounds for the bandwidth

3 New Bounds on Bandwidth

3.1 Bounds for $K_6 \times K_6 \times K_6$

The best bounds known for $bw\left(K_6^3\right)$ are

$$96 \leq bw\left(K_6^3\right) \leq 110.$$

3.1.1 Upper Bound

The best previous upper bound (from [7], reviewed in Section 1) is $4 \cdot 27 + (6/2) - 1 = 110$. This formula is a special case of that for $[n]^d$, n even, which was derived from the optimal numbering for $[2]^d$, and conjectured (in [7]) to be optimal for all n (even) and all d. The following tables give a numbering of $[6] \times [6] \times [6]$ (the vertex set of $K_6 \times K_6 \times K_6$) that has bandwidth 101:

1	2	9	28	65	89
3	4	10	29	67	90
13	14	15	30	69	107
37	38	39	43	71	132
77	79	81	83	163	175
91	92	101	124	164	189

5	6	11	31	66	93
7	8	12	32	68	94
16	17	18	33	70	108
40	41	42	44	72	141
78	80	82	84	165	176
95	96	102	131	166	194

19	20	23	34	73	115
21	22	24	35	74	116
25	26	27	36	75	123
45	46	47	48	76	144
117	119	121	134	173	201
118	120	122	135	174	202

49	50	53	58	145	146
51	52	54	59	151	152
55	56	57	60	155	156
61	62	63	64	159	160
147	149	153	157	205	206
148	150	154	158	207	208

85	86	103	125	161	185
87	88	104	127	167	188
109	110	111	129	169	197
133	137	139	142	171	199
177	179	181	183	209	210
186	187	195	203	211	212

97	98	105	126	162	190
99	100	106	128	168	191
112	113	114	130	170	198
136	138	140	143	172	200
178	180	182	184	213	214
192	193	196	204	215	216

This numbering was constructed to make the vertex-boundaries of its initial segments be as close as possible to the dynamic programming lower bound. The quotient Hamming balls of radius 0 minimize vertex-boundary for small v and those of radius 1 minimize for large v, so it starts off with subcubes (quotient Hamming balls of radius 0) up to $4 \times 4 \times 4$ (Note that $4/6 = 2/3$, which just happens to be the side of the subcube that minimizes maximum vertex boundary in $[0,1]^3$). The subcube then grows "arms" that eventually transform it into a quotient Hamming ball of radius 1. However, great care had to be taken in the process of interpolating between the two, to achieve the bandwidth of 101. Note that the numbering is not stable (*i.e.* unchanged by left-shifting) nor is it self-dual (isomorphic to its reverse numbering). It not only gives a better upper bound for $bw\,(K_6 \times K_6 \times K_6)$, but by the "blowing up" procedure in the proof of Theorem 1 it gives an upper bound of $\frac{101}{215}$ for $bw\,([0,1]^3)$. Actually, the argument in Theorem 1 gives a slightly larger upper bound of $\frac{101+1}{216} = \frac{102}{216}$, but if the same numbering is used recursively to "blow up" the subcubes, the upper bound, c, must satisfy the equation

$$c = \frac{101 + c}{216}$$

whose solution is $\frac{101}{215} = 0.469\,8$. This is considerably better than the previous best of .5 and halfway to the lower bound of .4439.

3.1.2 Lower Bound

Perhaps even more remarkable is that the numbering above can be shown to minimize the bandwidth of $K_6 \times K_6 \times K_6$. The solution of the vertex-isoperimetric problem on $[0,1]^3$ gives a lower bound of $\lceil (.4439) \times 216 \rceil = 96$ for $bw\left([6]^3\right)$. Solution of the discrete VIP, by generating all 1.48×10^{12} ideals in $[6] \times [6] \times [6]$ and evaluating their vertex boundaries, gives a lower bound of 100 for $bw\left(K_6^3\right)$. This calculation took about 6 days on a 2.5 Ghz PC, which made the VIP dynamic programming lower bound seem impractical to calculate. However, the lower bound of 101, which shows our upper bound to be sharp, was achieved in a calculation of about the same length by restricting the dynamic program to "good" ideals, those with $|\Phi|(S)| \leq 100$. This variant of the Branch and Bound strategy works beautifully because the requirement that ideals be "good" eliminates most of them from consideration just when the number of ideals (of cardinality k) becomes too large. The program showed that the longest chain of nested "good" ideals starting with $\varnothing$ terminates with $|S| = 51$. Thus every numbering, η, of $[6]^3$, which corresponds to a nested family of ideals, $\{S_k(\eta) : 0 \leq k \leq 216\}$, must have some k such that $|\Phi|(S_k(\eta))| > 100$ and so $bw\left([6]^3\right) \geq 101$.

3.2 A Nearly Sharp Numbering for $[0,1]^3$

Our numbering of $[6]^3$ decreased the known upper bound on $bw\left([6]^3\right)$ by a surprising amount, even decreasing the known bandwidth of $[0,1]^3$. This led us to try the same numbering strategy on $[0,1]^3$ directly. The result was the following numbering: First we fix two constants, $a, b \geq 0$ with $a + b \leq 1$. The values a and b will be determined at the end so as to optimize the result. To define the numbering, we partition $[0,1]^3$ into six, essentially disjoint, regions.

Region I: $[0,a]^3$.

Region II: $[0,a] \times [0,a] \times [a,a+b] \cup [0,a] \times [a,a+b] \times [0,a] \cup [a,a+b] \times [0,a] \times [0,a]$.

Region III: $[0,a] \times [0,a] \times [a+b,1] \cup [0,a] \times [a+b,1] \times [0,a] \cup [a+b,1] \times [0,a] \times [0,a]$

Region IV: $[0,a] \times [a,1] \times [a,a+b] \cup [0,a] \times [a,a+b] \times [a,1] \cup [a,a+b] \times [0,a] \times [a,1] \cup [a,a+b] \times [a,1] \times [0,a] \cup [a,1] \times [0,a] \times [a,a+b] \cup [a,1] \times [a,a+b] \times [0,a]$.

Region V: $[0,a] \times [a+b,1] \times [a+b,1] \cup [a+b,1] \times [0,a] \times [a+b,1] \cup [a+b,1] \times [a+b,1] \times [0,a]$.

Region VI: $[a,1]^3$.

Before putting any numbers into Region $i+1$, we completely fill Region i. Each Region is filled as follows:

Region I: The number t^3 will be assigned to any point, $(x, y, z) \in [0, a]^3$ with $\max\{x, y, z\} = t$. Thus the level surfaces of this function, in this Region, will be the faces of the subcube $[0, t]^3$.

Region II: We build up simultaneously the three faces of $[0, a]^3$, assigning value $a^3 + 3a^2t$ to any point $(x, y, a+t) \in [0, a] \times [0, a] \times [a, a+b]$, $(x, a+t, z) \in [0, a] \times [a, a+b] \times [0, a]$ or $(a + t, y, z) \in [a, a + b] \times [0, a] \times [0, a]$.

Region III: Again we symmetrically fill the three boxes. The points in $[0, a]\times[0, a]\times[a+b, 1]$ of the form (x, t, z) with $x \le t$ or (t, y, z) with $y \le t$ will be assigned value $a^3 + 3a^2b + 3(1-a-b)t^2$ and symmetrically for $[0, a] \times [a+b, 1] \times [0, a]$ and $[a+b, 1] \times [0, a] \times [0, a]$.

Region IV: Again we fill the six boxes symmetrically, but in this case there is the slight complication that the boxes are not pairwise disjoint. The points in $[0, a] \times [a, 1] \times [a, a + b]$ of the form (x, y, t) with $y \ge t$ will be assigned value $a^3 + 3a^2(1 - a) + 6(t - a)a(1 - a) - 3a(t - a)^2$.

Region V: Again we fill the three boxes symmetrically. The points of $[0, a] \times [a + b, 1] \times [a+b, 1]$ of the form (t, y, z) will be assigned value $a^3 + 3a^2(1-a) + 6ab(1-a) - 3ab^2 + 3(1 - a - b)^2t$.

Region VI: Here it does not really matter what the labeling is, but for the sake of consistency and clarity, we assign the point $(x, y, z) \in [a, 1]^3$, such that $t = \min\{x, y, z\}$, the value $1 - (1 - t)^3$.

Note the function, $f : [0, 1]^3 \to [0, 1]$ is defined to be monotone increasing and measure preserving.

Now we look at the calculation of upper bounds of the maximal differences between neighboring points. Neighboring pairs of points agree in two coordinates. From our requirement of monotonicity, we need only consider those pairs for which the one coordinate they differ in will have a 0 in one and a 1 in the other. Since no Region contains both members of such a pair and some pairs of Regions do not share such a pair, it suffices to check the maximum difference between the following pairs of Regions:

(i) Region I – Region III: Points corresponding to the same value of $t, 0 \le t \le a$, will have difference $[a^3+3a^2b+3(1-a-b)t^2]-t^3$. This function is maximized at $t = 2(1-a-b)$. If $2 \le 3a + 2b$ then it can be achieved, and the maximum is $a^3 + 3a^2b + 4(1 - a - b)^3$. Otherwise, $2 \ge 3a + 2b$ and the maximum value, at $t = a$, is $3a^2 - 3a^3$. Thus we have

$$f_1(a, b) = \begin{cases} a^3 + 3a^2b + 4(1 - a - b)^3 & \text{if } 2 \le 3a + 2b \\ 3a^2 - 3a^3 & \text{if } 2 \ge 3a + 2b \end{cases}.$$

(ii) Region II – Region IV: Points corresponding to the same value of $t, 0 \le t \le b$, will have difference $[a^3+3a^2(1-a)+6a(1-a)t-3at^2]-(a^3+3a^2t) = 6at+3a^2-3a^3-3at^2-9a^2t$. As a function of t this reaches its maximum when $t = (2-3a)/2$. If $(2-3a)/2 \le b$ (which is equivalent to $2 \le 3a+2b$), then the maximum value is $3a^2-3a^3+\frac{3}{4}(2-3a)^2$. Otherwise, $2 \ge 3a + 2b$ and the maximum, achieved at $t = b$, is $3a^2 - 3a^3 + 3a(2 - 3a)b - 3ab^2$. Thus we have

$$f_2(a, b) = \begin{cases} 3a^2 - 3a^3 + \frac{3}{4}(2 - 3a)^2 & \text{if } 2 \le 3a + 2b \\ 3a^2 - 3a^3 + 3a(2 - 3a)b - 3ab^2 & \text{if } 2 \ge 3a + 2b \end{cases}.$$

(iii) Region III – Region V: Points corresponding to the same value of t, $0 \le t \le a$, have difference $[a^3+3a^2b+3(1-a-b)t^2]-[a^3+3a^2(1-a)+6ab(1-a)-3ab^2+3(1-a-b)^2t] = 3a^2(1-a)+6ab(1-a)-3ab^2+3(1-a-b)^2t-3a^2b-3(1-a-b)t^2$. This reaches its maximum at $t=(1-a-b)/2 < a$, so its maximum is

$$f_3(a,b) = 3a^2 - 3a^3 + 6ab - 9a^2b - 3ab^2 + \frac{3}{4}(1-a-b)^3.$$

(iv) Region IV – Region VI: Points corresponding to the same value of $t, a \le t \le a+b$ will have difference

$$1-(1-t)^3 - [a^3 + 3a^2(1-a) + 6(t-a)a(1-a) - 3a(t-a)^2].$$

As a function of t this reaches its maximum when $t = 1-2a$. If $a \le 1-2a$ (which is equivalent to $a \le 1/3$), then the maximum value is $3a^3+3a^2-3a+1$. Otherwise, $a \ge 1/3$ and the maximum, achieved at $t=a$, is $3a^3-6a^2+3a$. Thus we have

$$f_4(a,b) = \begin{cases} 3a^3+3a^2-3a+1 & \text{if } a \le 1/3 \\ 3a^3-6a^2+3a & \text{if } a \ge 1/3 \end{cases}.$$

(v) Region V – Region VI: The maximum difference is between the least value of Region V and the largest of Region VI so

$$f_5(a,b) = (1-a)^3 + 3(1-(a+b))^2 a.$$

We wish to minimize the maximum of the five functions, f_1, f_2, f_3, f_4 and f_5. With the aid of Maple, we found that this min-max is assumed at $a = .6023, b = .1676$ where f_1, f_2, f_3 take the common value .4497, $f_4 = .285\,79$ and $f_5 = .158\,6$.

4 Comments and Conclusions

We have made progress on a fundamental mathematical problem, calculating $bw([0,1]^3)$. In [7] it was conjectured that the optimal numbering of $[2]^d$ showed how to optimize the bandwidth of $[0,1]^d$ for all d. In this paper we see that a weighted version of $[3]^d$ does better for $d=3$, decreasing the relative difference, $\underline{\Delta} = (\text{UB}/\text{VIP}) - 1$ from .1263 to .0131, a 90% reduction. If the dynamic programming lower bound can be proven to be $4/9 = .4444$, that would reduce $\underline{\Delta}$ by another 10%. However, not being able to achieve the precise determination of $bw([0,1]^3)$ that we had sought and anticipated has been frustrating. The same methods should work for $d = 4, 5$ *etc.*, lowering the known upper bounds, but seem unlikely to give exact calculations unless we can figure out how to do it in 3 dimensions.

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