Computational Completeness

1 Definitions and examples

Let $\Sigma = \{f_1, f_2, \ldots, f_i, \ldots\}$ be a (finite or infinite) set of Boolean functions. Any of the functions $f_i \in \Sigma$ can be a function of arbitrary number of arguments.

Definition 1 *The set Σ is called* **computationally complete** *(or, simply, complete), if any Boolean function can be expressed as a formula involving just the functions of the set Σ.*

Example 1 *The set $\Sigma_1 = \{\bar{x}_1,\ x_1 \vee x_2,\ x_1 \wedge x_2\}$ is complete, because any Boolean function can be represented in the SOP or in the POS form, and these representations involve just the functions of Σ_1.*

Example 2 *The set $\Sigma_2 = \{\bar{x}_1,\ x_1 \wedge x_2\}$ is complete, because $x_1 \vee x_2 = \overline{\bar{x}_1 \wedge \bar{x}_2}$. Therefore, the completeness of Σ_2 follows from the completeness of Σ_1.*

Example 3 *The set $\Sigma_3 = \{x_1|x_2\}$, where $x_1|x_2 = \overline{x_1 \wedge x_2}$, is complete. Indeed,*

$$x_1|x_1 = \bar{x}_1, \quad (x_1|x_2)|(x_1|x_2) = x_1 \wedge x_2.$$

Thus, the question concerning the completeness of Σ_3 is reduced to one of Σ_2.

Example 4 *The set $\Sigma_4 = \{1,\ x_1 \wedge x_2,\ x_1 \oplus x_2\}$, where $x_1 \oplus x_2$ is the XOR function and 1 is the constant function, is complete. Indeed, $x_1 \oplus 1 = \bar{x}$. Hence, the completeness of Σ_4 follows from the completeness of Σ_2.*

Example 5 *The set $\Sigma_5 = \{1,\ x_1 \wedge x_2\}$ is not complete, because any function that can be expressed by a formula involving just the functions of Σ_5 is either the constant function 1 or the function of the form $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ for $n = 2, 3, \ldots.$*

Given a set Σ of Boolean functions, how to recognize if Σ is complete ? In order to present a complete answer to this question we introduce 5 following classes of Boolean functions: T_0, T_1, L, S, and M.

2 The class T_0

Definition 2 *The class T_0 consists of all Boolean functions f (of any number of arguments) defined as follows:*

$$T_0 = \{f(x_1, \ldots, x_n) \mid f(0, 0, \ldots, 0) = 0\}.$$

Example 6 *The following functions belong to the class* T_0*:* 0, $x_1 \wedge x_2$, $x_1 \vee x_2$, $x_1 \oplus x_2$.

Example 7 *The functions* 1 *and* $\bar{x}_1$ *are not in* T_0*.*

The number of function in T_0 which depend on n variables is $2^{2^n-1} = \frac{1}{2} \cdot 2^{2^n}$.

3 The class T_1

Definition 3 *The class* T_1 *consists of all Boolean functions* f *(of any number of arguments) defined as follows:*

$$T_1 = \{f(x_1, \dots, x_n) \mid f(1, 1, \dots, 1) = 1\}.$$

Example 8 *The following functions belong to the class* T_1*:* 1, $x_1 \wedge x_2$, $x_1 \vee x_2$.

Example 9 *The functions* $x_1 \oplus x_2$ *and* $\bar{x}_1$ *are not in* T_1*.*

The number of function in T_1 which depend on n variables also equals $2^{2^n-1} = \frac{1}{2} \cdot 2^{2^n}$.

4 The class L of linear functions

Definition 4 *The class* L *consists of functions (of any number of arguments) that can be represented in the form*

$$L = \{f(x_1, \dots, x_n) \mid f = (a_1 \wedge x_1) \oplus (a_2 \wedge x_2) \oplus \cdots \oplus (a_n \wedge x_n) \oplus b\},$$

where $a_1, \dots, a_n, b \in \{0, 1\}$ *are some fixed constants.*

Example 10 *Since* $\bar{x}_1 = x_1 \oplus 1$*, then* $\bar{x}_1 \in L$*.*

Example 11 $x_1 \vee x_2 \notin L$*. Indeed, assume the contrary, i.e.* $x_1 \vee x_2 \in L$*. Then* $x_1 \vee x_2 = (a_1 \wedge x_1) \oplus (a_2 \wedge x_2) \oplus b$ *for some constants* $a_1, a_2, b \in \{0, 1\}$*. Since* $x_1 \vee x_2$ *significantly depends on two variables (i.e. cannot be represented as a function of one or less variables), then* $a_1 = a_2 = 1$*, because otherwise we would get a function depending on just one variable. Hence, our representation should be of the form* $x_1 \vee x_2 = x_1 \oplus x_2 \oplus b$ *for some* $b \in \{0, 1\}$*.*

Now if we assume $b = 0$*, then* $x_1 \vee x_2 = x_1 \oplus x_2$*, which is a contradiction. Otherwise, if* $b = 1$*, then* $x_1 \vee x_2 = x_1 \oplus x_2 \oplus 1 = \overline{x_1 \oplus x_2}$*, which is a contradiction too. The only way to avoid a contradiction is to accept* $x_1 \vee x_2 \notin L$*.*

Similarly $x_1 \wedge x_2 \notin L$*.*

The number of functions in L which depend of n variables is 2^{n+1}, because any such a function can be encoded by the binary vector $(a_1, a_2, \dots, a_n, b)$ consisting of $n + 1$ entries.

5 The class S of self-dual functions

Definition 5 *A Boolean function is called* **self-dual** *if*

$$f(x_1, \ldots, x_n) = \bar{f}(\bar{x}_1, \ldots, \bar{x}_n)$$

for any $x_1, \ldots, x_n \in \{0, 1\}$.

The class S consists of all self-dual Boolean functions (of any number of variables).

Example 12 *Obviously,* $\bar{x}_1 \in S$. *A more complicated example is the majority function*

$$f(x_1, x_2, x_3) = x_1x_2 \vee x_1x_3 \vee x_2x_3.$$

Indeed, using the DeMorgan's theorem

$$\begin{aligned}
\bar{f}(\bar{x}_1, \bar{x}_2, \bar{x}_3) &= \overline{\bar{x}_1\bar{x}_2 \vee \bar{x}_1\bar{x}_3 \vee \bar{x}_2\bar{x}_3} \\
&= \overline{\bar{x}_1\bar{x}_2} \vee \overline{\bar{x}_1\bar{x}_3} \vee \overline{\bar{x}_2\bar{x}_3} \\
&= (x_1 \vee x_2)(x_1 \vee x_3)(x_2 \vee x_3) \\
&= x_1x_2 \vee x_1x_3 \vee x_2x_3 \\
&= f(x_1, x_2, x_3).
\end{aligned}$$

Example 13 *The functions* $f_1(x_1, x_2) = x_1 \wedge x_2$ *and* $f_2(x_1, x_2) = x_1 \vee x_2$ *do not belong to* S. *Indeed,* $f_1(0, 1) = f_1(1, 0)$ *and* $f_2(0, 1) = f_2(1, 0)$.

What is the number of the self-dual functions depending on n variables ? To compute this number, represent the function $f(x_1, \ldots, x_n) \in S$ by the logical table with 2^n rows, which we split into two equal parts, consisting of 2^{n-1} rows each:

$x_1\ x_2\ \cdots\ x_n$	$f(x_1, \ldots, x_n)$
$0\ 0 \cdots 0\ 0$	α
$0\ 0 \cdots 0\ 1$	
$\cdots$	
$0\ 1 \cdots 1\ 1$	
$1\ 0 \cdots 0\ 0$	
$1\ 0 \cdots 0\ 1$	
$\cdots$	
$1\ 1 \cdots 1\ 1$	$\bar{\alpha}$

Note that the i^{th} row of the left part of the table is the negation of the $(2^n - i)^{th}$ row. If $f(x_1, \ldots, x_n) \in S$ then the value of f in these rows are different. Therefore, f is completely determined by the values it takes on just in the upper (or just lower) part of the table. In other words, the number of the self-dual functions in question equals the number of binary strings of length 2^{n-1}, i.e. $2^{2^{n-1}} = \sqrt{2^{2^n}}$.

6 The class M of monotone functions

Definition 6 *Let* $(x_1, \ldots, x_n)$ *and* $(y_1, \ldots, y_n)$ *be two binary vectors of the same dimension. We write* $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ *if* $x_i \leq y_i$ *for* $i = 1, 2, \ldots, n$.

If $(x_1, \ldots, x_n) \not\leq (y_1, \ldots, y_n)$ *and* $(y_1, \ldots, y_n) \not\leq (x_1, \ldots, x_n)$ *then we say that these vectors are* **incompatible**.

Example 14 *It holds:* $(0, 1, 0) \leq (1, 1, 0)$, *and* $(0, 0, \ldots, 0) \leq (1, 1, \ldots, 1)$.

Example 15 *The vectors* $(0, 1, 0)$ *and* $(1, 0, 0)$ *are incompatible. In general, a vector and its binary coordinatewise negation are incompatible, cf. e.g.* $(0, 1, 0)$ *and* $(1, 0, 1)$.

Definition 7 *We call a function* $f(x_1, \ldots, x_n)$ **monotone** *if* $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$ *whenever* $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$.

Example 16 *The functions* $x_1 \wedge x_2$ *and* $x_2 \vee x_2$ *are monotone, however the functions* $\bar{x_1}$ *and* $x_1 \oplus x_2$ *are not.*

Denote by M_n the number of monotone Boolean functions of n variables. The problem of computing M_n was posed by Dedekind in 1897 (!) and is still unsolved up to now. It is known that

n	M_n
0	2
1	3
2	6
3	20
4	168
5	7581
6	7828354

Many mathematicians contributed to this problem. The most recent to our knowledge result (cf. [1, 4]) is the asymptotic formula for M_n as $n \to \infty$:

$$M_n \sim \begin{cases} 2^{\binom{n}{n/2}} \exp\left\{\binom{n}{n/2-1}\left(2^{-n/2} + n^2 \cdot 2^{-n-5} - n \cdot 2^{-n-4}\right)\right\}, & \text{if } n \text{ is even} \\ 2 \cdot 2^{\binom{n}{(n-1)/2}} \exp\left\{\binom{n}{(n+1)/2}\left(2^{-(n+1)/2} + n^2 \cdot 2^{-n-4}\right) + \right. \\ \left. +\binom{n}{(n-3)/2}\left(2^{-(n+3)/2} - n^2 \cdot 2^{-n-6} - n \cdot 2^{-n-3}\right)\right\}, & \text{if } n \text{ is odd.} \end{cases}$$

7 The criterion for completeness

Let $\Sigma = \{f_1, f_2, \ldots, f_i, \ldots\}$ be a set of Boolean functions.

Theorem 1 (E. Post [2, 3])
The set Σ is complete if and only if for any of the classes T_0, T_1, L, S, M there exists a function of Σ which is not in this class.

In order to apply this theorem to the set Σ we construct the following table:

	T_0	T_1	L	S	M
f_1					
f_2					
$\ldots$					
f_i					
$\ldots$					

with entries of the set $\{+, -\}$. The entry "+" in the i^{th} row means that the function f_i belongs to the corresponding class. Then, by the theorem of Post, the set Σ is complete if and only if each column of this table contains at least one "–".

Example 17 *Consider the system $\Sigma_1 = \{\bar{x}_1,\ x_1 \vee x_2,\ x_1 \wedge x_2\}$. One has:*

	T_0	T_1	L	S	M
$\bar{x}_1$	$-$	$-$	$+$	$+$	$-$
$x_1 \vee x_2$	$+$	$+$	$-$	$-$	$+$
$x_1 \wedge x_2$	$+$	$+$	$-$	$-$	$+$

Thus, by the theorem of Post the set Σ is complete.

Moreover, one of the last two functions (but not both) can be deleted from Σ without the lost of the completeness of the remaining set. In such a way the complete system Σ_2 of Example 2 can be obtained.

Example 18 *Consider the following set Σ:*

$$f_1 = x_1 x_2,\ f_2 = 0,\ f_3 = 1,\ f_4 = x_1 \oplus x_2 \oplus x_3.$$

One has

	T_0	T_1	L	S	M
f_1	$+$	$+$	$-$	$-$	$+$
f_2	$+$	$-$	$+$	$-$	$+$
f_3	$-$	$+$	$+$	$-$	$+$
f_4	$+$	$+$	$+$	$+$	$-$

Thus, Σ is complete. However, deleting of any function from σ makes the remaining set incomplete because

$$\begin{array}{ll} \{f_2, f_3, f_4\} \subset L & \{f_1, f_3, f_4\} \subset T_1 \\ \{f_1, f_2, f_4\} \subset T_0 & \{f_1, f_2, f_3\} \subset M \end{array}$$

Corollary 1 *Any complete set Σ of functions contains a complete subset consisting of at most 5 functions of Σ.*

In fact a more strong result holds: any complete set can be reduced to a complete subset consisting of at most 4 functions. As Example 18 shows, this proposition cannot be further improved in general.

References

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[4] A.A. Sapozhenko *On the number of antichains in multileveled ranked sets*, (in Russian), Diskretnaya Matematika, vol. 1 (1989), No. 2, 110–128.