Mathematical Background

1. The O-calcul
2. Some basic sums
3. Recurrences and the Master Theorem
    - Applications of the Master Method
    - Proof of the Master Method

The O-calcul

Definition 1 *For functions $f(n)$ and $g(n)$, and integer n we write (following E. Landau):*

$$f(n) = o(g(n)) \qquad \iff \qquad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Example 1

$$n^\alpha = o(n^\beta) \quad \iff \quad \alpha < \beta.$$

The Relation $o(\cdot)$ is transitive, that is

$$f(n) = o(g(n)) \quad \wedge \quad g(n) = o(h(n)) \quad \Rightarrow \quad f(n) = o(h(n)),$$

...and provides a classification of functions, e.g.:

$$1 = o(\log\log n), \qquad \log\log n = o(\log n), \qquad \log n = o(n^\epsilon)$$

$$n^\epsilon = o(n^c), \qquad n^c = o(n^{\log n}), \qquad n^{\log n} = o(c^n)$$

$$c^n = o(n^n), \qquad n^n = o(c^{c^n}),$$

where $0 < \epsilon < 1 < c$.

This classification only makes sense for large n.

Example 2 *Consider $\log n = o(n^{0.0001})$ and let $n = 10^{100}$. One has: $\log n = 100$ but $n^{0.0001} \approx 1.023$.*

On the other hand, let $n = 10^{10^{100}}$. Then: $\log n = 10^{100}$ comparing to $n^{0.0001} = 10^{10^{96}}$.

Definition 2 *For functions $f(n)$ and $g(n)$, and integer n we write (following P. Bachmann, 1894):*

$$f(n) \;=\; O(g(n)) \quad \text{for } n \to \infty \qquad \Longleftrightarrow$$
$$\exists n_0 \, \forall n \geq n_0 : \; |f(n)| \;\leq\; c|g(n)|$$

for some constant c.

Example 3

$$4n^3 + 5n^2 - 6n + 2 = O(n^3),$$

but also

$$4n^3 + 5n^2 - 6n + 2 = O(n^{10}).$$

Definition 3 *We write:*

$$f(n) = \Omega(g(n)) \quad \Longleftrightarrow \quad \forall n \geq n_0 : \; |f(n)| \geq c|g(n)|$$

for some constant $c > 0$.

Obviously, it holds:

$$f(n) = \Omega(g(n)) \quad \Longleftrightarrow \quad g(n) = O(f(n)).$$

Definition 4 *We write for $n \to \infty$:*

$$\begin{aligned} f(n) \;&=\; \Theta(g(n)) \quad \Longleftrightarrow \\ f(n) \;&=\; O(g(n)) \quad \wedge \quad f(n) = \Omega(g(n)). \end{aligned}$$

Definition 5 *We write for $n \to \infty$:*

$$\begin{aligned} f(n) \sim g(n) \quad &\Longleftrightarrow \quad f(n) = g(n) + o(g(n)) \\ &\Longleftrightarrow \quad \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1. \end{aligned}$$

It holds:

$$\begin{aligned}
n^m &= O(n^{m'}), \qquad \text{for } m \le m' \\
O(f(n)) + O(g(n)) &= O(|f(n)| + |g(n)|), \\
f(n) &= O(f(n)), \\
c \cdot O(f(n)) &= O(f(n)), \quad \text{if } c = \text{const} \\
O(O(f(n))) &= O(f(n)), \\
O(f(n)) \cdot O(g(n)) &= O(f(n) \cdot g(n)) = f(n) \cdot O(g(n)), \\
\log(1 + O(f(n))) &= O(f(n)), \qquad \text{if } f(n) = o(1) \\
e^{O(f(n))} &= 1 + O(f(n)), \quad \text{if } f(n) = o(1).
\end{aligned}$$

Example 4 *Assume* $|x| < 1$. *One has:*

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + O(x^4), \\
\ln(1+x) &= x - \frac{x^2}{2} + O(x^3), \\
\frac{1}{1-x} &= 1 + x + x^2 + O(x^3),
\end{aligned}$$

Stirling's Formula:

$$\begin{aligned}
n! &= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right), \\
\binom{n}{k} &\sim \frac{n^k}{k!}, \qquad \text{if } k = o(\sqrt{n}), \\
\binom{n}{\lfloor \frac{n}{2} \rfloor} &\sim \sqrt{\frac{2}{\pi n}}\, 2^n.
\end{aligned}$$

Basic Sums

- Arithmetic series:

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1) = \binom{n+1}{2} = \Theta(n^2)$$

- Geometric series:

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1} \qquad \text{for } x \neq 1$$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad \text{for } |x| < 1$$

- Harmonic series:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1)$$

where $\gamma \approx 0.577$ is the Euler's Constant.

- Telescopic series:

$$\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0$$

$$\sum_{k=0}^{n-1} (a_k - a_{k+1}) = a_0 - a_n$$

Example 5

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n}$$

Theorem 1 [1] (Master Method)
Let $a \geq 1$ and $b > 1$ be constants, and let $T(n)$ be an integer variable function defined by

$$T(n) = a \cdot T(n/b) + f(n)$$

(here n/b is either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$). One has:

a. If $f(n) = O(n^{\log_b a - \epsilon})$ for a constant $\epsilon > 0$, then

$$T(n) = \Theta(n^{\log_b a})$$

b. If $f(n) = \Theta(n^{\log_b a})$, then

$$T(n) = \Theta(n^{\log_b a} \log n)$$

c. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and $\forall n \geq n_0 :$ $af(n/b) \leq c \cdot f(n)$ for some constant $c < 1$, then

$$T(n) = \Theta(f(n))$$

The Master Method is applicable **not for all** functions $f(n)$!
e.g. **not** for $f(n) = n^{\log_b a} \log n$.

Applications of the Master Method

Let

$$T(n) = 9T(n/3) + n.$$

For $a = 9$ and $b = 3$ we get

$$n^{\log_b a} = n^{\log_3 9} = n^2$$

and

$$f(n) = n = O(n^{\log_3 9 - \epsilon})$$

for $\epsilon = 1$. Therefore, the conditions of case (a.) are satisfied, and

$$T(n) = \Theta(n^2).$$

Let

$$T(n) = T(2n/3) + 1.$$

Since $a = 1$ and $b = 3/2$, we have

$$n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$$

and

$$f(n) = 1 = \Theta(n^{\log_b a}) = \Theta(1).$$

Case (b.) provides

$$T(n) = \Theta(\log n).$$

Let

$$T(n) = 3T(n/4) + n\log n.$$

For $a = 3$ and $b = 4$ we have

$$n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})$$

and

$$f(n) = n\log n = \Omega(n^{\log_4 3+\epsilon})$$

for $\epsilon \approx 0.2$. So, the condition of case (c.) are satisfied.

Regularity condition of case (c.):

$$af(n/b) = 3(n/4)\log(n/4) \le 0.75 \cdot n\log n = 0.75 \cdot f(n)$$

for n sufficiently large. One has

$$T(n) = \Theta(n\log n).$$

If, however,

$$T(n) = 2T(n/2) + n\log n$$

one has

$$n^{\log_b a} = n^{\log_2 2} = n,$$

and

$$\frac{f(n)}{n^{\log_b a}} = \frac{n\log n}{n} = \log n = o(n^{\epsilon})$$

for any $\epsilon > 0$.

Therefore, the Master Method in this case is not applicable.

Proof of the Master Theorem

We give a proof only for the case when n is an exact power of b.

Lemma 1 *Let $a \geq 1$ and $b > 1$ be constants, and let $f(n)$ be a nonnegative function defined on* **exact** *powers of b. Define*

$$T(n) = \begin{cases} \Theta(1), & \text{if } n = 1 \\ aT(n/b) + f(n), & \text{if } n = b^i \end{cases}$$

where i is a positive integer. Then

$$T(n) = \Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j).$$

Proof. Applying the recursion for n/b, n/b^2, ..., we get

$$\begin{aligned} T(n) &= aT(n/b) + f(n) \\ &= a(aT(n/b^2) + f(n/b)) + f(n) \\ &= a^2 T(n/b^2) + af(n/b) + f(n) \\ &= a^2(aT(n/b^3) + f(n/b^2)) + af(n/b) + f(n) \\ &= a^3 T(n/b^3) + a^2 f(n/b^2) + af(n/b) + f(n) \\ &= \cdots \\ &= a^k T(n/b^k) + \sum_{j=0}^{k-1} a^j f(n/b^j) \end{aligned}$$

For $k = \log_b n$ one has $T(n/b^k) = \Theta(1)$ and $a^k = a^{\log_b n} = n^{\log_n a \log_b n} = n^{\frac{\log_b n}{\log_a n}} = n^{\log_b a}$, so

$$T(n) = n^{\log_b a}\Theta(1) + \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j) \qquad \square$$

Hence, the solution $T(n)$ depends on $g(n) = \Sigma_{j=0}^{\log_b n-1} a^j f(n/b^j)$.

The proof is split in 3 cases depending on the function $f(n)$ as in the Master Theorem.

Lemma 2 *Under the assumptions of Lemma 1, if*

$$f(n) = O(n^{\log_b a-\epsilon})$$

for some constant $\epsilon > 0$*, then*

$$g(n) = O(n^{\log_b a}).$$

Proof. One has $f(n/b^j) = O((n/b^j)^{\log_b a-\epsilon})$, so

$$\begin{aligned}
g(n) &= O\left(\sum_{j=0}^{\log_b n-1} a^j \left(\frac{n}{b^j}\right)^{\log_b a-\epsilon}\right) \\
&= O\left(n^{\log_b a-\epsilon} \sum_{j=0}^{\log_b n-1} \left(\frac{ab^\epsilon}{b^{\log_b a}}\right)^j\right) \\
&= O\left(n^{\log_b a-\epsilon} \sum_{j=0}^{\log_b n-1} (b^\epsilon)^j\right) \\
&= O\left(n^{\log_b a-\epsilon} \left(\frac{b^{\epsilon \log_b n-1}}{b^\epsilon - 1}\right)\right) \\
&= O\left(n^{\log_b a-\epsilon} \left(\frac{n^\epsilon - 1}{b^\epsilon - 1}\right)\right) \\
&= O\left(n^{\log_b a-\epsilon} O(n^\epsilon)\right) \\
&= O(n^{\log_b a}) \qquad \square
\end{aligned}$$

Lemma 3 *Under the assumptions of Lemma 1, if*

$$f(n) = \Theta(n^{\log_b a})$$

then

$$g(n) = \Theta(n^{\log_b a} \lg n)$$

Proof. We have $f(n/b^j) = \Theta((n/b^j)^{\log_b a})$, so

$$\begin{aligned} g(n) &= \Theta\left(\sum_{j=0}^{\log_b n - 1} a^j \left(\frac{n}{b^j}\right)^{\log_b a}\right) \\ &= \Theta\left(n^{\log_b a} \sum_{j=0}^{\log_b n - 1} \left(\frac{a}{b^{\log_b a}}\right)^j\right) \\ &= \Theta\left(n^{\log_b a} \sum_{j=0}^{\log_b n - 1} 1\right) \\ &= \Theta\left(n^{\log_b a} \log_b n\right) \\ &= \Theta\left(n^{\log_b a} \lg n\right) \qquad \square \end{aligned}$$

Lemma 4 *Under the assumptions of Lemma 1, if*

$$af(n/b) \le cf(n)$$

for some constant $c < 1$ *and for all* $n \ge b$*, then*

$$g(n) = \Theta(f(n))$$

Proof. Note that $af(n/b) \le cf(n) \iff f(n/b) \le (c/a)f(n)$. Iterating this inequality j times, we get $f(n/b^j) \le (c/a)^j f(n)$, or, equivalently, $a^j f(n/b^j) \le c^j f(n)$.

Therefore,

$$\begin{aligned} g(n) &= \sum_{j=0}^{\log_b n-1} a^j\, f(n/b^j) \\ &\le \sum_{j=0}^{\log_b n-1} c^j\, f(n) \\ &\le f(n) \sum_{j=0}^{\infty} c^j \\ &= f(n) \left(\frac{1}{1-c}\right) \\ &= O((f(n)) \end{aligned}$$

On the other hand, since all terms in the sum for $g(n)$ are non-negative, $g(n) = \Omega(f(n))$. $\square$

Using the estimates for $g(n)$ given in Lemmas 2 - 4, we can prove the Master Theorem in the case n is an exact power of b.

Proof. (of Master Theorem)
By Lemma 1, $T(n) = \Theta(n^{\log_b a}) + g(n)$.

- if $f(n) = O(n^{\log_b a-\epsilon})$ for some constant $\epsilon > 0$, then using Lemma 2,

$$\begin{aligned} T(n) &= \Theta(n^{\log_b a}) + O(n^{\log_b a}) \\ &= \Theta(n^{\log_b a}) \end{aligned}$$

- if $f(n) = \Theta(n^{\log_b a})$, then using Lemma 3,

$$\begin{aligned} T(n) &= \Theta(n^{\log_b a}) + \Theta(n^{\log_b a} \lg n) \\ &= \Theta(n^{\log_b a} \lg n) \end{aligned}$$

- if $f(n) = \Omega(n^{\log_b a+\epsilon})$ for some constant $\epsilon > 0$ and if $af(n/b) \leq cf(n)$ for some constant $c < 1$, then by Lemma 4,

$$\begin{aligned} T(n) &= \Theta(n^{\log_b a}) + \Theta(f(n)) \\ &= \Theta(f(n)) \end{aligned}$$

  because $f(n) = \Omega(n^{\log_b a+\epsilon})$. □

The Master Theorem is also true if the floors ane ceilings are used in the master recurrence.