CHAPTER 4
Decidability

Outline

- **Decidable Languages**
    - **decidable problems concerning regular languages**
    - **decidable problems concerning context-free languages**
- The Halting Problem
    - The diagonalization method
    - The halting problem is undecidable
    - A Turing unrecognizable languages

Decidability (intro.)

- We have introduced Turing machines as a model of a general purpose computer
- We defined the notion of algorithm in terms of Turing machines by means of the Church-Turing thesis
- In this chapter we
  - begin to investigate the power of algorithms to solve problems
  - demonstrate certain problems that can be solved algorithmically and others that cannot
- Our objective is to explore the limits of algorithmic solvability
- Why should we study unsolvability? Showing that a problem is unsolvable doesn't appear to be of any use if we have to solve it. But …
- We need to study this phenomenon for two reasons:
  - First, knowing that a problem is algorithmically unsolvable is useful because then you realize that the problem must be simplified or altered before you can find an algorithmic solution.
  - The second reason is cultural. Even if you deal with problems that clearly are solvable, a glimpse of the unsolvable can stimulate your imagination and help you gain an important perspective on computation.

Decidable Languages

• In this section we give some examples of languages that are decidable by algorithms.

• For example, we present an algorithm which tests whether a string is a member of a context-free language.

• This problem is related to the problem of recognizing and compiling programs in a programming language.

Decidable Problems Concerning Regular Languages

• We begin with certain computation problems concerning finite automata

• We give algorithms for testing whether a finite automata accepts a string, whether the language of a finite automaton is empty, and whether two finite automata are equivalent.

• For convenience we use languages to represent various computational problems.

• For example, the ***acceptance problem*** for DFAs of testing whether a particular finite automaton accepts a given string can be expressed as a language, A_{DFA} .

$$A_{DFA} = \{< B, w >: B \textit{ is a DFA that accepts input string } w\}.$$

• The problem of testing whether a DFA *B* accepts an input *w* is the same as the problem of testing whether $<B,w>$ is a member of the language A_{DFA} .

• Similarly, we can formulate other computational problems in terms of testing membership in a language. Showing that a language is decidable is the same as showing that the computation problem is decidable (= algorithmically solvable).

The Acceptance Problem for DFAs is Decidable

Theorem 1 A_{DFA} is a decidable language.

- We present a TM *M* that decides A_{DFA} .

M = "on input <*B,w*>, where *B* is a DFA and *w* is a string:

1. Simulate *B* on input *w.*
2. If the simulation ends in an accept state, *accept*. If it ends in a non-accepting state, *reject.* "

A few implementation details:

- The input is <*B,w*>. It is a representation of a DFA *B* together with a string *w*. One reasonable representation of *B* is a list of its five components, $Q, \Sigma, \delta, q_0, F$.
- When *M* receives its input, *M* first checks on whether it properly represents a DFA *B* and a string *w*. If not, it *rejects*.
- Then *M* carries out the simulation in a direct way. It keeps track of *B*'s current state and *B*'s current position in the input *w.*
- Initially, *B*'s current state is q_0 and *B*'s current position is the leftmost symbol of *w.*
- The states and position are updated according to the specified transition function δ.
- When *M* finishes processing the last symbol of *w*, *M accepts* if *B* is in an accepting state; *M rejects* if *B* is in a non-accepting state.

The Acceptance Problem for NFAs and REXs.

We can prove similar result for NFAs and Regular Expressions.

$$A_{NFA} = \{ <B, w>: B \text{ is a NFA that accepts input string } w \}.$$

Theorem 2: A_{NFA} is a decidable language.

N = "on input <*B,w*>, where *B* is a NFA and *w* is a string:

1. Convert NFA *B* to an equivalent DFA *C* using the procedure for this conversion given in Theorem "subset construction".
2. Run TM *M* from Theorem 1 on input <*C,w*>.
3. If *M* accepts, *accept*, otherwise *reject*."

Running TM *M* in stage 2 means incorporating *M* into the design of *N* as a subprocedure.

$$A_{REX} = \{ <R, w>: R \text{ is a regular Expression that generates string } w \}.$$

Theorem 3: A_{REX} is a decidable language.

P = "on input <*R,w*>, where *R* is a reg.expr. and *w* is a string:

1. Convert *R* to an equivalent DFA *C* using the procedure for this conversion given in Theorem earlier.
2. Run TM *M* from Theorem 1 on input <*C,w*>.
3. If *M* accepts, *accept*, otherwise *reject*."

The Emptiness Problem for the Language of a Finite Automaton.

$$E_{DFA} = \{<A>: A \text{ is a DFA and } L(A) = \varnothing\}.$$

Theorem 4: E_{DFA} is a decidable language.

• A DFA accepts some string if and only if reaching an accept state from the start state by traveling along the arrows of the DFA is possible.

• To test this condition we can design a TM *T* that uses marking algorithm similar to that used in example "*connectedness of a graph*".

T = "on input <*A*>, where *A* is a DFA :

1. Mark the start state of *A*.
2. Repeat the following stage until no new states get marked:
3. Mark any state that has a transition coming into it from any state that is already marked.
4. If no accept state is marked, *accept*; otherwise *reject*."

The Equivalence Problem for Finite Automata.

$$EQ_{DFA} = \{< A, B >: A \text{ and } B \text{ are } DFAs \text{ and } L(A) = L(B)\}.$$

Theorem 5: EQ_{DFA} is a decidable language.

• Consider a symmetric difference of *L(A)* and *L(B),* i.e a language *L(C)*

$$L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right).$$

The complement of *L(A)*

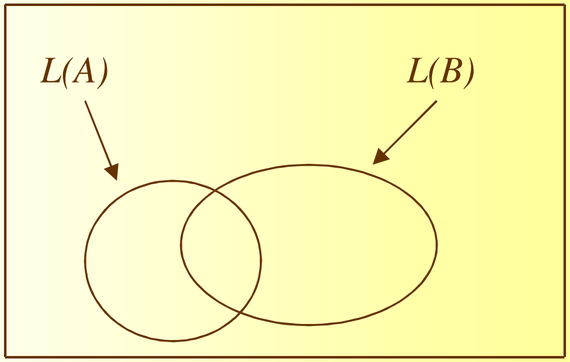
• Hence, $L(C) = \varnothing \text{ if and only if } L(A) = L(B).$

• We can construct *C* from *A* and *B* with the constructions for proving the class of regular languages closed under complementation, union, and intersection.

• These constructions are algorithms that can be carried out by Turing machines.

F= “on input <*A,B*>*,* where *A,B* are DFAs :

1. Construct DFA *C* as described.
2. Run TM *T* from theorem 4 on input <*C*>.
3. If *T* accepts, *accept*; if *T* rejects, *reject*.”



Decidable Problems Concerning CFLs

- Here we describe algorithms to test whether a CFG generates a particular string and to test whether the language of a CFG is empty.
- Let $A_{CFG} = \{<G, w>: G \text{ is a CFG that generates string } w\}.$

Theorem 6: A_{CFG} is a decidable language.

- For CFG *G* and string *w* we want to test whether *G* generates *w*.
- One idea is to use *G* to go through all derivations to determine whether any is a derivation of *w*. This idea doesn't work, as infinitely many derivations may have to be tried. If *G* does not generate *w*, this algorithm would never halt. Hence this idea gives a TM which is *recognizer*, not a *decider*.
- To make this TM into a decider we need to ensure that the algorithm tries only finite many derivations.
- If *G* is in Chomsky normal form, any derivation of *w* has *2n-1* steps, where *n* is the length of *w*. Only finite many such derivations exist.
- We present a TM *S* that decides A_{CFG} .

S = "on input *<G,w>,* where *G* is a CFG and *w* is a string:

1. Convert *G* to an equivalent grammar in Chomsky normal form.
2. List all derivations with *2n-1* steps, where *n* is the length of *w*, except if *n=0*, then instead list all derivations with 1 step.
3. If any of these derivations generate *w*, *accept*; if not, *reject.* "

Decidable Problems Concerning CFLs(cont.)

- Here we describe an algorithm to test whether the language of a CFG is empty.
- Let $E_{CFG} = \{<G>: G \text{ is a CFG and } L(G) = \emptyset\}.$

Theorem 7: E_{CFG} is a decidable language.

- For CFG *G* we need to test whether the start variable can generate a string of terminals.
- The algorithm does so by solving a more general problem. It determines for each variable whether that variable is capable of generating a string of terminals.
- When the algorithm has determined that a variable can generate some string of terminals, the algorithm keeps track of this information by placing a mark on that variable. First the algorithm marks all terminal symbols in the grammar.
- Then it scans all the rules of the grammar. If it ever finds a rule that permits some variable to be replaced by some string of symbols all of which are already marked, the algorithm knows that this variable can be marked, too.
- The algorithm continues in this way until it cannot mark any additional variables. The TM *R* implements this algorithm.

R = "on input <*G*>, where *G* is a CFG:

1. Mark all terminals in *G*. Repeat (2) until no new variables get marked:
2. Mark any variable *A* where *G* has a rule $A \rightarrow U_1U_2...U_k$ and each symbol $U_1, U_2, ..., U_k$ has already been marked.
3. If the start symbol is not marked, *accept*; otherwise *reject*. "

Decidable Problems Concerning CFLs(cont.)

- Let $EQ_{CFG} = \{<G,H>: G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$.
- This language is ***undecidable*** (we cannot apply technique used in " EQ_{DFA} is decidable"; the class of CFLs is not closed under complementation and intersection).

- We can prove now the following.
- ***Theorem 8***: Every CFL is decidable.
- Let *A* be a CFL and *G* be a CFG for *A*.

- Here is a TM *M(G)* that decides *A*.
- We build a copy of *G* into *M(G).*
- *S* is a TM from Theorem 6.

M(G) = "on input *w:*

1. Run TM *S* on input <*G,w*>
2. If this machine accepts, *accept*; if it rejects, *reject.* "