Outline

8.1 Arc Length

Suppose a curve C is defined by equation $y = f(x)$, where $f(x)$ is continuous on $[a, b]$. We subdivide $[a, b]$ with points $x_0 = a, x_1, x_2, \ldots, x_{n-1}, x_n = b$ into equal-size intervals.

If $y_i = f(x_i)$ then the point $P_i = (x_i, y_i)$ is on C.

Definition

The length L of C is defined as

$$\boxed{L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|}$$

For $\Delta y_i = y_i - y_{i-1}$ and $\Delta x = x_i - x_{i-1}$ we get

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

By the Mean Value Theorem, for some $x_i^* \in [x_{i-1}, x_i]$ it holds

$$\Delta y_i = f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1}) = f'(x_i^*)\Delta x$$

Thus,

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x)^2 + (\Delta y_i)^2} = \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} \\ &= \sqrt{1 + [f'(x_i^*)]^2} \cdot \sqrt{(\Delta x)^2} = \sqrt{1 + [f'(x_i^*)]^2} \cdot \Delta x \end{aligned}$$

Therefore, if $f(x)$ is continuous on $[a, b]$, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2} \cdot \Delta x = \boxed{L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx}$$

Example

Find the length of the curve $y^2 = x^3$ between $(1,1)$ and $(4,8)$.
We have

$$y = x^{3/2} \qquad \frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

So, the length is

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

Substituting $u = 1 + (9/4)x$, $du = (9/4)x\ dx$, we get

$$\begin{aligned} L &= \frac{4}{9}\int_{13/4}^{10} \sqrt{u}\, du = \frac{4}{9}\cdot\frac{2}{3}u^{3/2}\Big]_{13/4}^{10} \\ &= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right] = \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right) \end{aligned}$$

If the curve is $x = g(y)$ and $g'(y)$ is continuous on $[c, d]$ then

$$\boxed{L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}$$

Example

Find the length of $y^2 = x$ from $(0, 0)$ to $(1, 1)$. We have

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

We use trigonometric substitution $y = \frac{1}{2}\tan\theta$ with $dy = \frac{1}{2}\sec^2\theta$ and $\sqrt{1 + 4y^2} = \sqrt{1 + \tan^2\theta} = \sec\theta$. For $y = 0$, $\tan\theta = 0$, so $\theta = 0$. For $y = 1$, $\tan\theta = 2$, so $\theta = \tan^{-1} 2 = \alpha$.

Putting all together we get

$$
\begin{aligned}
L &= \int_0^{\alpha} \sec\theta \cdot \frac{1}{2}\sec^2\theta \, d\theta = \frac{1}{2}\int_0^{\alpha} \sec^3\theta \, d\theta \\
&= \frac{1}{4}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^{\alpha} \\
&= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)
\end{aligned}
$$

For $\tan\alpha = 2$ we have $\sec^2\alpha = 1 + \tan^2\alpha = 5$, so $\sec\alpha = \sqrt{5}$.

Therefore,

$$
L = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5}+2)}{4}
$$

The Arc Length Function

Definition

For a smooth curve $y = f(x)$ on $[a, b]$ let $s(x)$ denote the arc length from $(a, f(a))$ to $(x, f(x))$ (the arc length function)

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

Example

Find the arc length function for $y = x^2 - \frac{1}{8} \ln x$ from $(1, 1)$.

$$\begin{aligned}
f'(x) &= 2x - \frac{1}{8x} \\
1 + [f(x)]^2 &= 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2} \\
&= 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2
\end{aligned}$$

So, $\sqrt{1+[f'(x)]^2} = 2x + \frac{1}{8x}$ and the arc length is

$$\begin{aligned} s(x) &= \int_1^x \sqrt{1+[f'(t)]^2}\, dt \\ &= \int_1^x \left(2t + \frac{1}{8t}\right)\, dt = t^2 + \frac{1}{8}\ln t\Big]_1^x \\ &= x^2 + \frac{1}{8}\ln x - 1 \end{aligned}$$

In particular, the arc length from $(1,1)$ to $(3, f(3))$ is

$$s(3) = 3^2 + \frac{1}{8}\ln 3 - 1 = 8 + \frac{\ln 3}{8}$$

Why $s(x) < 0$ for $x < 1$?

8.2 Area of a Surface of Revolution

Cutting a circular cone with base radius r and slant length ℓ results in a plain circle sector with radius ℓ and central angle $\theta = 2\pi r/\ell$. Its area is

$$A = \frac{1}{2}\ell^2\theta = \frac{1}{2}\ell^2\left(\frac{2\pi r}{\ell}\right) = \pi r\ell$$

Similarly, the area of a frustum of a cone with radii r_1 and r_2 and slant length ℓ can be found as

$$A = 2\pi\ell\frac{r_1 + r_2}{2}$$

In general, if a curve $y = f(x)$ is rotating about the x-axis, we approximate it by line segments and approximate the surface of its rotation as the sum of cone areas

$$2\pi\frac{y_{i-1} + y_i}{2}|P_{i-1}P_i| = 2\pi\frac{y_{i-1} + y_i}{2}\sqrt{1 + [f'(x_i^*)]^2}$$

Since f is continuous, $y_{i-1} \approx y_i \approx f(x_i^*)$, so

$$A \approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

This leads to the formulas (for $a \le x \le b$, $c \le y \le d$)

$$\boxed{S = 2\pi \int_a^b y\sqrt{1 + (y')^2}\, dx \quad \text{(rotation of } y(x) \text{ about the } x\text{−axis)}}$$

$$\boxed{S = 2\pi \int_c^d y\sqrt{1 + (x')^2}\, dy \quad \text{(rotation of } x(y) \text{ about the } x\text{−axis)}}$$

The formulas for computing the area or rotation about the y-axis are similar (for $a \le x \le b$, $c \le y \le d$):

$$S = 2\pi \int_a^b x\sqrt{1 + (y')^2}\, dx \quad \text{(rotation of } y(x) \text{ about the } y\text{−axis)}$$

$$S = 2\pi \int_c^d x\sqrt{1 + (x')^2}\, dy \quad \text{(rotation of } x(y) \text{ about the } y\text{−axis)}$$

Example

Find the area of the surface obtained by rotating the curve $y = \sqrt{4-x^2}$, $-1 \le x \le 1$ about the x-axis.

$$\begin{aligned} S &= \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_{-1}^{1} 2\pi \sqrt{4-x^2} \sqrt{1 + \frac{x^2}{4-x^2}}\, dx \\ &= 2\pi \int_{-1}^{1} \sqrt{4-x^2} \frac{2}{\sqrt{4-x^2}}\, dx \\ &= 4\pi \int_{-1}^{1} 1\, dx = 4\pi(2) \\ &= 8\pi \end{aligned}$$

Example

The arc of parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface.

$$
\begin{aligned}
S &= \int_1^2 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= 2\pi \int_1^2 x\sqrt{1 + 4x^2}\, dx \\
&= \frac{\pi}{4} \int_5^{17} \sqrt{u}\, du \qquad (u = 1 + 4x^2, \;\; du = 8x\, dx) \\
&= \frac{\pi}{6} u\sqrt{u}|_5^{17} \\
&= \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})
\end{aligned}
$$

Alternative solution: use the inverse function $x = \sqrt{y}$ and rotate it about the x-axis:

$$
\begin{aligned}
S &= \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}^{2} \right)}\, dy \\
&= 2\pi \int_1^4 \sqrt{y} \sqrt{1 + \frac{1}{4y}}\, dy \\
&= \pi \int_1^4 \sqrt{4y+1}\, dy \\
&= \frac{\pi}{4} \int_5^{17} \sqrt{u}\, du \qquad (u = 1 + 4y) \\
&= \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5})
\end{aligned}
$$

Example

Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the x-axis.

$$\begin{aligned}
S &= \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx \\
&= 2\pi \int_0^1 \sqrt{1 + u^2}\, du \qquad (u = e^x) \\
&= 2\pi \int_{\pi/4}^{\alpha} \sec^3 \theta\, d\theta \qquad (u = \tan \theta,\ \alpha = \tan^{-1} e) \\
&= \pi[\sec \theta \tan \theta + \ln|\sec \theta + \tan \theta|]_{\pi/4}^{\alpha} \\
&= \pi[\sec \alpha \tan \alpha + \ln(\sec \alpha + \tan \alpha) - \sqrt{2} - \ln(\sqrt{2} + 1) \\
&= \pi[e\sqrt{1 + e^2} + \ln(e + \sqrt{1 + e^2}) - \sqrt{2} - \ln(\sqrt{2} + 1)
\end{aligned}$$

(since $\tan \alpha = e$, we have $\sec \alpha = \sqrt{1 + e^2}$)

Example

Find the area of the surface obtained by rotating the curve $x = \frac{1}{3}(y^2+2)^{3/2}$, $1 \le y \le 2$ about the x-axis.

Rewrite the equation in the form $3x = (y^2+2)^{3/2}$, from where we get $y = \sqrt{(3x)^{2/3}-2}$.

Then $y' = \frac{1}{2\sqrt{(3x)^{2/3}-2}} \cdot \frac{2}{3} \cdot \frac{3}{(3x)^{1/3}} = \frac{1}{\sqrt{(3x)^{2/3}-2} \cdot (3x)^{1/3}}$ and

$$\begin{aligned} 1+(y')^2 &= 1 + \frac{1}{((3x)^{2/3}-2)(3x)^{2/3}} \\ &= \frac{((3x)^{2/3})^2 - 2\cdot(3x)^{2/3}+1}{((3x)^{2/3}-2)(3x)^{2/3}} \\ &= \frac{((3x)^{2/3}-1)^2}{((3x)^{2/3}-2)(3x)^{2/3}} \end{aligned}$$

Therefore,

$$\begin{aligned}
S &= \int_1^{\sqrt{8}} 2\pi\sqrt{(3x)^{2/3}-2}\,\frac{(3x)^{2/3}-1}{\sqrt{(3x)^{2/3}-2)}\cdot(3x)^{1/3}}\,dx \\
&= \int_1^{\sqrt{8}} 2\pi\frac{(3x)^{2/3}-1}{(3x)^{1/3}}\,dx \\
&= 2\pi\int_1^{\sqrt{8}} ((3x)^{2/3}-1)\cdot\frac{3}{2}\cdot\frac{1}{3}\,d\left((3x)^{2/3}-1\right) \\
&= \pi\int_{\sqrt[3]{9}-1}^{2\sqrt[3]{9}-1} u\,du \qquad \left(u=(3x)^{2/3}-1\right) \\
&= \ldots \quad \text{the rest is just arithmetic}
\end{aligned}$$

Alternative solution: $x' = \frac{1}{3} \cdot \frac{3}{2}\sqrt{y^2+2} \cdot 2y = y\sqrt{y^2+2}$.
Hence, $1 + (x')^2 = y^2(y^2+2) + 1 = (y^2+1)^2$. So,

$$\begin{aligned}
S &= 2\pi \int_1^2 y\sqrt{1+(x')^2}\,dy \\
&= 2\pi \int_1^2 y(y^2+1)\,dy \\
&= 2\pi \int_1^2 (y^2+1)\,d\left(\frac{y^2+1}{2}\right) \\
&= \frac{\pi}{2}\,(y^2+1)^2\Big]_1^2 \\
&= \frac{\pi}{2}(25-4) \\
&= \frac{21\pi}{2}
\end{aligned}$$

8.3 Applications to Physics and Engineering

Hydrostatic Pressure and Force

If a plain surface of area A is submerged into a fluid at depth d, then the fluid above the area has volume $V = Ad$ and weight $m = \rho V = \rho Ad$. The force exerted by the fluid and the pressure on the plain is then

$$F = mg = \rho gAd \qquad P = \frac{F}{A} = \rho gd$$

where $g = 9.81 m/s^2$ is the gravitation constant. The density of water $\rho = 1000 kg/m^3$.

Important principle: at any point in a liquid the pressure is the same in all directions.

Moments and Centers of Mass

If masses $m_1, m_2, \ldots, m_n$ are located at points $x_1, x_2, \ldots, x_n$ on a line, then the **moment of the system about the origin** and the **center of mass** of the system are defined as

$$M = \sum_{i=1}^{n} m_i x_i$$

$$\overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

For a 2-dim area we define the moments about the x- and y-axes M_x and M_y in a similar way. The center of mass is the point $(\overline{x}, \overline{y})$ with

$$\overline{x} = \frac{M_x}{m}, \quad \overline{y} = \frac{M_y}{m} \qquad \text{where } m = \sum_{i=1}^{n} m_i$$

For a plane plate with density ρ bounded by a smooth curve $y = f(x)$ on $[a, b]$, we subdivide the interval $[a, b]$ with points $x_1, \ldots, x_n$ on subintervals of equal length Δx. This splits the area below the curve into rectangles. The centroid of the i-th rectangle is $C_i = \left(\overline{x_i}, \frac{1}{2} f(\overline{x_i})\right)$, where $\overline{x_i} = (x_{i-1} + x_i)/2$.

The rectangle area and mass is $f(\overline{x_i})\Delta x$ and $\rho f(\overline{x_i})\Delta x$. The moment of the rectangle R_i is then

$$M_y(R_i) = [\rho f(\overline{x_i})\Delta x]\overline{x_i} = \rho \overline{x_i} f(\overline{x_i})\Delta x$$

By letting $n \to \infty$ we derive the following formula

$$\boxed{M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \overline{x_i} f(\overline{x_i})\Delta x = \rho \int_a^b x f(x)\, dx}$$

Similarly, the moment of R_i about the x-axis is

$$M_x(R_i) = [\rho f(\overline{x_i})\Delta x]\frac{1}{2}f(\overline{x_i}) = \frac{\rho}{2}[f(\overline{x_i})]^2\Delta x$$

and

$$\boxed{M_x = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\rho}{2}[f(\overline{x_i})]^2\Delta x = \frac{\rho}{2}\int_a^b [f(x)]^2\,dx}$$

Taking into account that the mass of the area is $m = \rho A = \rho\int_a^b f(x)\,dx$ we get

$$\begin{aligned}\overline{x} &= \frac{M_y}{m} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx}\\ \overline{y} &= \frac{M_x}{m} = \frac{\frac{1}{2}\int_a^b [f(x)]^2\,dx}{\int_a^b f(x)\,dx}\end{aligned}$$

Example

Find the center of mass of a semicircular plate of radius r.
By putting the origin into the circle center we have $f(x) = \sqrt{r^2 - x^2}$. By symmetry, $\overline{x} = 0$.

$$\begin{aligned}
\overline{y} &= \frac{1}{2A}\int_{-r}^{r}[f(x)]^2\,dx \\
&= \frac{1}{\frac{1}{2}\pi r^2}\cdot\frac{1}{2}\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2\,dx \\
&= \frac{2}{\pi r^2}\int_{0}^{r}(r^2 - x^2)\,dx \\
&= \frac{2}{\pi r^2}\left[r^2 x - \frac{x^3}{3}\right]_0^r \\
&= \frac{2}{\pi r^2}\frac{2r^3}{3} = \frac{4r}{3\pi}
\end{aligned}$$

Example

Find the centroid of the area bounded by the curves $y = \cos x$, $y = 0$, $x = 0$, $x = \pi/2$.

The area of the plate is $\int_0^{\pi/2} \cos x \; dx = 1$. We get

$$\begin{aligned}
\overline{x} &= \frac{1}{A}\int_0^{\pi/2} x f(x)\; dx = \int_0^{\pi/2} x \cos x \; dx \\
&= x \sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x \; dx \\
&= \frac{\pi}{2} - 1 \\
\overline{y} &= \frac{1}{2A}\int_0^{\pi/2} [f(x)]^2 \; dx = \frac{1}{2}\int_0^{\pi/2} \cos^2 x \; dx \\
&= \frac{1}{4}\int_0^{\pi/2} (1 + \cos 2x)\; dx = \frac{1}{4}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} \\
&= \frac{\pi}{8}
\end{aligned}$$

A similar approach works for computing the centroid of a region between two curves $f(x)$ and $g(x)$. The formulas become:

$$\boxed{\begin{aligned} \overline{x} &= \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx \\ \overline{y} &= \frac{1}{2A}\int_a^b \left([f(x)]^2 - [g(x)]^2\right)\,dx \end{aligned}}$$

Example

Find the centroid of the region bounded by $y = x$ and $y = x^2$.

The curves intersect at points $x = 0$ and $x = 1$. We have

$$A = \int_0^1 (x - x^2)\,dx = \left.\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

Therefore,

$$
\begin{aligned}
\overline{x} &= \frac{1}{A}\int_0^1 x[f(x) - g(x)]\,dx = 6\int_0^1 x(x - x^2)\,dx \\
&= 6\int_0^1 (x^2 - x^3)\,dx \\
&= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 \\
&= \frac{1}{2} \\
\overline{y} &= \frac{1}{2A}\int_0^1 ([f(x)]^2 - [g(x)]^2)\,dx = 3\int_0^1 (x^2 - x^4)\,dx \\
&= 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 \\
&= \frac{2}{5}
\end{aligned}
$$

Theorem (of Pappus)

Let R be a plane region that lies entirely on one side of a line ℓ in the plane. If R is rotated about ℓ, then the volume of the resulting solid is the product of the area A of R and the distance d traveled by the centroid of R.

The proof is for a special case when R lies between curves $f(x)$ and $g(x)$ and ℓ is the y-axis. We have

$$\begin{aligned} V &= 2\pi \int_a^b x[f(x) - g(x)]\, dx \qquad \text{(see Section 5.3)} \\ &= 2\pi(\bar{x}A) = (2\pi\bar{x})A = dA \end{aligned}$$

Example

Find the volume of the torus obtained by rotating a circle of radius r, that is at distance R ($R > r$) from the center, about the y-axis.

$$V = dA = (2\pi R)(\pi r^2) = 2\pi r^2 R$$

8.4 Applications to Biology

Blood Flow

The law of laminar flow

$$v(r) = \frac{P}{4\eta\ell}(R^2 - r^2)$$

gives the velocity of blood that flows along a blood vessel with radius R and length ℓ at a distance r from the central axis.

We compute the rate of blood flow (volume per unit time) by splitting the vessel section in concentric circles of equally spaced radii $r_1, r_2, \ldots, r_n$. The approximate area of a washer of outer radius r_i and width Δr is $2\pi r_i \Delta r$. The flow of blood across the washer section is then

$$(2\pi r_i \Delta r) v(r_i) = 2\pi r_i v(r_i) \Delta r$$

The total volume of blood across the entire vessel section is

$$\sum_{i=1}^{n} 2\pi r_i v(r_i) \Delta r$$

For the total amount of flux we have

$$\begin{aligned} F &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r_i\, v(r_i)\Delta r = \int_0^R 2\pi r\, v(r)\, dr \\ &= \int_0^R 2\pi r \frac{P}{4\eta\ell}(R^2 - r^2)\, dr \\ &= 2\pi \frac{P}{4\eta\ell} \int_0^R (R^2 r - r^3)\, dr = \frac{\pi P}{2\eta\ell}\left[R^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^R \\ &= \frac{\pi P}{2\eta\ell}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{\pi P R^4}{8\eta\ell} \end{aligned}$$

This is Poiseuille's Law.