Outline

7.1 Integration by Parts

The product rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

implies

$$\int f(x)g'(x)dx + \int f'(x)g(x)dx = f(x)g(x)$$

In other terms,

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx}$$

Example

Find $\int x \sin x \, dx$.

For $f(x) = x$ and $g'(x) = \sin x$ we have $f'(x) = 1$ and $g(x) = -\cos x$, so

$$\begin{aligned}
\int x \sin x \, dx &= f(x)g(x) - \int g(x)f'(x) \, dx \\
&= x(-\cos x) - \int (-\cos x) dx \\
&= -x \cos x + \int \cos x \, dx \\
&= -x \cos x + \sin x + C
\end{aligned}$$

For $u = f(x)$ and $v = g(x)$ one has $du = f'(x)\,dx$
$dv = g'(x)\,dx$, so the formula for integration by parts becomes

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example

$$\begin{aligned} u &= x & dv &= \sin x\,dx \\ du &= dx & v &= -\cos x \end{aligned}$$

$$\begin{aligned} \int x \sin x\,dx &= \int \overbrace{x}^{u}\,\overbrace{\sin x\,dx}^{dv} = \overbrace{x}^{u}\,\overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v}\,\overbrace{dx}^{du} \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + C \end{aligned}$$

Example

Find $\int \ln x \, dx$. For this let

$$\begin{aligned} u &= \ln x & dv &= dx \\ du &= \frac{1}{x} dx & v &= x \end{aligned}$$

One has

$$\begin{aligned} \int \ln x \, dx &= uv - \int v \, du \\ &= x \ln x - \int x \frac{dx}{x} \\ &= x \ln x - \int dx \\ &= x \ln x - x + C \end{aligned}$$

Example

Find $\int t^2 e^t \, dt$. For this let

$$\begin{array}{rclrcl} u &=& t^2 & dv &=& e^t \, dt \\ du &=& 2t \, dt & v &=& e^t, \quad \text{hence} \end{array}$$

$$\int t^2 e^t \, dt = uv - \int v \, du = t^2 e^t - 2 \int t e^t \, dt$$

Apply integration by parts again with

$$\begin{array}{rclrcl} u &=& t & dv &=& e^t \, dt \\ du &=& dt & v &=& e^t, \quad \text{so} \end{array}$$

$$\begin{aligned} \int t^2 e^t \, dt &= uv - \int v \, du = t^2 e^t - 2 \int t e^t \, dt \\ &= t^2 e^t - 2(t e^t - e^t + C) \\ &= t^2 e^t - 2 t e^t + 2 e^t + C_1 \end{aligned}$$

Example

Compute $\int e^x \sin x \, dx$. Let $u = e^x$ and $dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$. We apply integration by parts twice:

$$\begin{aligned}\int e^x \sin x \, dx &= -e^x \cos x + \int e^x \cos x \, dx \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx\end{aligned}$$

Hence,

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + C$$

and

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

Integration by parts also applies to definite integrals:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\,dx$$

Example

Calculate $\int_0^1 \tan^{-1} x\,dx$. Let $u = \tan^{-1} x$, $dv = dx$. Then $du = \frac{dx}{1+x^2}$ and $v = x$.

$$\begin{aligned}
\int_0^1 \tan^{-1} x\,dx &= x\tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}dx \\
&= 1\cdot\tan^{-1} 1 - 0\cdot\tan^{-1} 0 - \int_0^1 \frac{x}{1+x^2}dx \\
&= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx
\end{aligned}$$

To evaluate the last integral we use substitution $t = 1 + x^2$, which implies

$$\begin{aligned}\int_0^1 \frac{x}{1+x^2}dx &= \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}\ln|t|\Big]_1^2 \\ &= \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2\end{aligned}$$

Putting all together,

$$\int_0^1 \tan^{-1} x \; dx = \frac{\pi}{4} - \frac{\ln 2}{2}$$

Example

Prove the reduction formula for an integer $n \geq 2$

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

Let $u = \sin^{n-1} x$, $dv = \sin x \, dx$. Then $v = -\cos x$ and $du = (n-1) \sin^{n-2} x \cos x \, dx$. Using $\cos^2 x = 1 - \sin^2 x$ implies

$$\begin{aligned} \int \sin^n x \, dx &= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx \\ &= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx \\ &\quad -(n-1) \int \sin^n x \, dx \end{aligned}$$

Hence,

$$n \int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx$$

7.2 Trigonometric Integrals

Strategy for taking integrals of the form $\int \sin^m x \cos^{2k+1} x \, dx$:

$$\begin{aligned} \int \sin^m x \cos^{2k+1} x \, dx &= \int \sin^m x (\cos^2 x)^k \cos x \, dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx \end{aligned}$$

and then substitute $u = \sin x$.

Example

$$\begin{aligned} \int \cos^3 x \, dx &= \int \cos^2 x \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx \\ &= \int (1 - u^2) du = u - \frac{1}{3} u^3 + C \\ &= \sin x - \frac{1}{3} \sin^3 x + C \end{aligned}$$

Strategy for taking integrals of the form $\int \sin^{2k+1} x \cos^n x \, dx$:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x \, dx &= \int (\sin^2 x)^k \cos^n x \sin x \, dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx\end{aligned}$$

and then substitute $u = \cos x$.

Example

$$\begin{aligned}\int \sin^5 x \cos^2 x \, dx &= \int (\sin^2 x)^2 \cos^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx \\ &= -\int (1 - u^2)^2 u^2 du = -\int (u^2 - 2u^4 + u^6) du \\ &= -\left(\frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7}\right) + C, \qquad u = \cos x\end{aligned}$$

Strategy for taking integrals of the form $\int \sin^{2m} x \cos^{2n} x \, dx$:

use half-angle identities $\sin x \cos x = \frac{1}{2} \sin 2x$ and

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Example

$$\begin{aligned}
\int_0^{\pi} \sin^2 x \, dx &= \frac{1}{2} \int_0^{\pi} (1 - \cos 2x) \, dx \\
&= \left[\frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) \right]_0^{\pi} \\
&= \frac{1}{2} \left(\pi - \frac{1}{2} \sin 2\pi \right) - \frac{1}{2} \left(0 - \frac{1}{2} \sin 0 \right) = \frac{\pi}{2}
\end{aligned}$$

Strategy for taking integrals of the form $\int \tan^m x \sec^{2k} x \, dx$: If $k \geq 2$ use $\sec^2 x = 1 + \tan^2 x$ to rewrite $\int$ in terms of $\tan x$.

Example

$$\begin{aligned}
\int \tan^6 x \sec^4 x \, dx &= \int \tan^6 x \sec^2 x \sec^2 x \, dx \\
&= \int \tan^6 x (1 + \tan^2 x) \sec^2 x \, dx \\
&= \int u^6 (1 + u^2) \, du \qquad (u = \tan x) \\
&= \int (u^6 + u^8) \, du = \frac{u^7}{7} + \frac{u^9}{9} + C \\
&= \frac{1}{7} \tan^7 x + \frac{1}{9} \tan^9 x + C
\end{aligned}$$

Strategy for taking integrals of the form $\int \tan^{2k+1} x \sec^n x \, dx$:
use $\tan^2 x = \sec^2 x - 1$ to rewrite $\int$ in terms of $\sec x$.

Example

$$\begin{aligned}
\int \tan^5 x \sec^7 x \, dx &= \int \tan^4 x \sec^6 x \sec x \tan x \, dx \\
&= \int (\sec^2 x - 1)^2 \sec^6 x \sec x \tan x \, dx \\
&= \int (u^2 - 1)^2 u^6 \, du \qquad (u = \sec x) \\
&= \int (u^{10} - 2u^8 + u^6) \, du \\
&= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C \qquad (u = \sec x)
\end{aligned}$$

In other cases there are no general methods. But it is helpful to know:

$$\boxed{\int \tan x \, dx = \ln|\sec x| + C}$$

$$\boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C}$$

Example

$$\begin{aligned}
\int \tan^3 x \, dx &= \int \tan x \tan^2 x \, dx = \int \tan x(\sec^2 x - 1) dx \\
&= \int \tan x \sec^2 x \, dx - \int \tan x \, dx \\
&= \frac{\tan^2 x}{2} - \ln|\sec x| + C
\end{aligned}$$

Example

With $u = \sec x$ and $dv = \sec^2 x \; dx$ one has $du = \sec x \tan x \; dx$ and $v = \tan x$. Integration by parts provides:

$$\begin{aligned} \int \sec^3 x \; dx &= \sec x \tan x - \int \sec x \tan^2 x \; dx \\ &= \sec x \tan x - \int \sec x (\sec^2 x - 1) \; dx \\ &= \sec x \tan x - \int \sec^3 x \; dx + \int \sec x \; dx \end{aligned}$$

Finally,

$$\int \sec^3 x \; dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$$

For taking integrals $\int \sin mx \cos nx \; dx$, $\int \sin mx \sin nx \; dx$ or $\int \cos mx \cos nx \; dx$ use identities:

- $\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Example

$$\begin{aligned}
\int \sin 4x \cos 5x \; dx &= \int \frac{1}{2}[\sin(-x) + \sin 9x] \; dx \\
&= \frac{1}{2} \int (-\sin x + \sin 9x) \; dx \\
&= \frac{1}{2}\left(\cos x - \frac{1}{9}\cos 9x\right) + C
\end{aligned}$$

7.3 Trigonometric Substitution

The regular substitution rule can be reversed as follows:

$$\int f(x)\,dx = \int f(g(t))g'(t)\,dt$$

Here is a set of helpful trig substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

For example, the expression $\sqrt{a^2 - x^2}$ becomes then

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = a|\cos\theta|$$

Example

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$. Let $x = 3\sin\theta$. Then $dx = 3\cos\theta\,d\theta$, so $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$.

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\,dx &= \int \frac{3\cos\theta}{9\sin^2\theta} 3\cos\theta\,d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta} d\theta = \int \cot^2\theta\,d\theta \\
&= \int (\csc^2\theta - 1)\,d\theta \\
&= -\cot\theta - \theta + C
\end{aligned}$$

Since $\sin\theta = x/3$, $\theta = \sin^{-1}(x/3)$ and $\cot\theta = \frac{\sqrt{9-x^2}}{x}$. Hence,

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Solving the equation for y for $x \geq 0$ and $y \geq 0$ results $y = \frac{b}{a}\sqrt{a^2 - x^2}$, so $A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$. Substitute $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$, so $\sqrt{a^2 - x^2} = a\cos\theta$ and

$$\begin{aligned} A &= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta \\ &= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta = 4ab\int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\,d\theta \\ &= 2ab\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left(\frac{\pi}{2} + 0 - 0\right) = \pi ab \end{aligned}$$

Example

Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$. Let $x = 2\tan\theta$, then $dx = 2\sec^2\theta\; d\theta$ and $\sqrt{x^2+4} = \sqrt{4(\tan^2\theta+1)} = \sqrt{4\sec^2\theta} = 2\sec\theta$.

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+4}}\,dx &= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{du}{u^2}\\ &= -\frac{1}{4u} + C = -\frac{1}{4\sin\theta} + C\end{aligned}$$

(here $u = \sin\theta$). Since $\tan\theta = x/2$, $\sin\theta = \frac{x}{\sqrt{x^2+4}}$, so

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = -\frac{\sqrt{x^2+4}}{4x} + C$$

Example

Find $\int \frac{dx}{\sqrt{x^2 - a^2}}$ for $a > 0$. Let $x = a\sec\theta$, then $dx = a\sec\theta\tan\theta \; d\theta$ and

$$\sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = a\tan\theta$$

Therefore,

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 - a^2}} &= \int \frac{a\sec\theta\tan\theta}{a\tan\theta}\, d\theta = \int \sec\theta \; d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \end{aligned}$$

Since $\sec\theta = x/a$, $\tan\theta = \sqrt{x^2 - a^2}/a$, and

$$\begin{aligned} \int \frac{dx}{\sqrt{x^2 - a^2}} &= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C \\ &= \ln|x + \sqrt{x^2 - a^2}| - \ln a + C \end{aligned}$$

Example

Find $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$. Substitute $x = \frac{3}{2}\tan\theta$ with $d\theta = \frac{3}{2}\sec^2\theta d\theta$ and $\sqrt{4x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta$. One has

$$
\begin{aligned}
\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx &= \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\frac{3}{2}\sec^2\theta\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\,d\theta
\end{aligned}
$$

Substitute $u = \cos\theta$, so $du = -\sin\theta\, d\theta$. One has

$$\begin{aligned}\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\, dx &= -\frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\, du \\ &= \frac{3}{16}\int_1^{1/2} (1-u^{-2})\, du = \frac{3}{16}\left[u + \frac{1}{u}\right]_1^{1/2} \\ &= \frac{3}{16}\left[\left(\frac{1}{2}+2\right) - (1+1)\right] = \frac{3}{32}\end{aligned}$$

Example

Find $\int \frac{x}{\sqrt{x^2+4}}\, dx$. With direct substitution $u = x^2 + 4$ ($du = 2x\, dx$) we get

$$\int \frac{x}{\sqrt{x^2+4}}\, dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+4} + C$$

Example

Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$. Our approach it to complete the square under the root sign: $3-2x-x^2 = 4-(x+1)^2$. Then substitute $x = u-1$:

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx = \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx = \int \frac{u-1}{\sqrt{4-u^2}}\,dx$$

Now let $u = 2\sin\theta$: $du = 2\cos\theta\,d\theta$, $\sqrt{4-u^2} = 2\cos\theta$

$$\begin{aligned}
\int \frac{u-1}{\sqrt{4-u^2}}\,dx &= \int \frac{2\sin\theta-1}{2\cos\theta}\,2\cos\theta\,d\theta \\
&= \int (2\sin\theta-1)\,d\theta = -2\cos\theta-\theta+C \\
&= -\sqrt{4-u^2}-\sin^{-1}(u/2)+C \\
&= -\sqrt{3-2x-x^2}-\sin^{-1}\frac{x+1}{2}+C
\end{aligned}$$

7.4 Integration of Rational Functions

Rational function is a ratio of polynomials. We express it as a sum of simpler functions, called *partial functions*, e.g.

$$\begin{aligned} \int \frac{x+5}{x^2+x-2}\, dx &= \int \left(\frac{2}{x-1} - \frac{1}{x+2} \right) dx \\ &= 2\ln|x-1| - \ln|x-2| + C \end{aligned}$$

We express a rational function $f(x) = \frac{P(x)}{Q(x)}$ as a sum of simpler fractions provided $\deg(P) < \deg(Q)$ (proper function). If $f(x)$ is improper we use long division to write it as

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} \quad \text{where} \quad \deg(R) < \deg(Q).$$

For example,

$$\frac{x^3+x}{x-1} = (x^2+x+2) + \frac{2}{x-1}$$

Next step is to factor $Q(x)$ as a product of linear polynomials and (maybe) irreducible quadratic factors (this is always possible!).

$$Q(x) = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

Third step is to express the proper rational function $P(x)/Q(x)$ as a sum of partial functions of the form

$$\frac{A}{(ax + b)^i} \quad \text{or} \quad \frac{Ax + B}{(ax^2 + bx + 1)^i}$$

Case 1: $Q(x)$ is a product of distinct linear factors

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_k x + b_k)$$

Then

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_k}{a_k x + b_k}$$

Example

Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$. The denominator is

$$2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$$

So, we are looking for A, B, C such that

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

Bring both parts to a common denominator:

$$\begin{aligned} x^2+2x-1 &= A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1) \\ &= (2A+B+2C)x^2 + (3A+2B-C)x - 2A \end{aligned}$$

We get a system of linear equations to determine A, B, C

$$\begin{array}{rcrcrcr} 2A & + & B & + & 2C & = & 1 \\ 3A & + & 2B & - & C & = & 2 \\ -2A & & & & & = & -1 \end{array}$$

from where we get $A = 1/2$, $B = 1/5$, and $C = -1/10$, so

$$\begin{aligned} \int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx &= \int \left[\frac{1}{2x} + \frac{1}{5(2x-1)} - \frac{1}{10(x+2)}\right] dx \\ &= \frac{\ln|x|}{2} + \frac{\ln|2x-1|}{10} - \frac{\ln|x+2|}{10} + C \end{aligned}$$

Example

Find $\int \frac{dx}{x^2-a^2}$ for $a \neq 0$. By method of partial fractions:

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

Hence,

$$A(x+a)+B(x-a)=1$$

For $x=a$ we get $A(2a)=1$, so $A=1/(2a)$.
Also, for $x=-a$ we get $B(-2a)=1$, so $B=-1/(2a)$.

Therefore

$$\begin{aligned}
\int \frac{dx}{x^2-a^2} &= \frac{1}{2a}\int \left(\frac{1}{x-a}-\frac{1}{x+a}\right)\, dx \\
&= \frac{1}{2a}(\ln|x-a|-\ln|x+a|)+C \\
&= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C
\end{aligned}$$

Case 2: $Q(x)$ contains a repeated linear factor

If some linear factor $(a_1 x + b_1)$ repeats r times, instead of the single term $A_1/(a_1 x + b_1)$ we use

$$\frac{A_1}{a_1 x + b_1} + \frac{A_2}{(a_1 x + b_1)^2} + \cdots + \frac{A_r}{(a_1 x + b_1)^r}$$

For example:

$$\frac{1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

Example

Find $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$.

The first step is to obtain a proper rational function:

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = (x+1) + \frac{4x}{x^3 - x^2 - x + 1}$$

The second step is to factorize $Q(x)$:

$$\begin{aligned} x^3 - x^2 - x + 1 &= (x-1)(x^2-1) = (x-1)(x-1)(x+1) \\ &= (x-1)^2(x+1) \end{aligned}$$

The fractional decomposition would be of the form

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

Multiplying it by the least common denominator $(x-1)^2(x+1)$

$$\begin{aligned} 4x &= A(x-1)(x+1) + B(x+1) + C(x-1)^2 \\ &= (A+C)x^2 + (B-2C)x + (-A+B+C) \end{aligned}$$

This way we get a linear system to find A, B, C:

$$\begin{array}{rcrcrcl} A & & & + & C & = & 0 \\ & & B & - & 2C & = & 4 \\ -A & + & B & + & C & = & 0 \end{array}$$

from where $A = 1$, $B = 2$, and $C = -1$ follows. Hence

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx = \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx$$

$$
\begin{aligned}
&= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C \\
&= \frac{x^2}{2} + x + -\frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + C
\end{aligned}
$$

Case 3: $Q(x)$ contains irreducible distinct quadratic factors

If the irreducible factor is $ax^2 + bx + c$ then we get the term of the form $(Ax + B)/(ax^2 + bx + c)$. For example,

$$
\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}
$$

The quadratic terms can be integrated by completing the square (if needed) and using the formula

$$\boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C}$$

Example

Evaluate $\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$. First, rewrite the function as

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying both parts by $x(x^2 + 4)$ results in

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x = (A + B)x^2 + Cx + 4A$$

which implies $A = 1$, $B = 1$, and $C = -1$, so

$$\begin{aligned}\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx &= \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)\,dx \\ &= \int \frac{dx}{x} + \int \frac{x\,dx}{x^2+4} - \int \frac{dx}{x^2+4} \\ &= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + C\end{aligned}$$

To evaluate the second integral in the middle line we used substitution $u = x^2 + 4$ with $du = 2x\,dx$.

Case 4: $Q(x)$ contains repeated irreducible quadratic factor

If some factor $ax^2 + bx + c$ repeats r times we get for it the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

For example,

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} +$$

$$+\frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

Example

Find $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\, dx$. The partial decomposition is

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

If we equate coefficients we get the system

$$A+B=0 \quad C=-1 \quad 2B+B+D=2 \quad C+E=-1 \quad A=1$$

... whose solution is $A = D = 1$, $B = C = -1$, and $E = 0$.

Putting all together,

$$\begin{aligned} &\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\, dx = \int \left(\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2} \right) dx \\ = &\int \frac{dx}{x} - \int \frac{x\, dx}{x^2+1} - \int \frac{dx}{x^2+1} + \int \frac{x\, dx}{(x^2+1)^2} \\ = &\ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1} x - \frac{1}{2(x^2+1)} + C \end{aligned}$$

Rationalizing substitutions

Some non-rational functions having expression of the form $\sqrt[n]{g(x)}$ can be rationalized by using substitution $u = \sqrt[n]{g(x)}$.

Example

Find $\int \frac{\sqrt{x+4}}{x}\,dx$. Let $u = \sqrt{x+4}$, so $x = u^2 - 4$, $dx = 2u\,du$.

$$\begin{aligned}
\int \frac{\sqrt{x+4}}{x}\,dx &= \int \frac{u}{u^2-4}\,2u\,du = 2\int \frac{u^2}{u^2-4}\,du \\
&= 2\int \left(1 + \frac{4}{u^2-4}\right)\,du = 2\int du + 8\int \frac{du}{u^2-4} \\
&= 2u + 8 \cdot \frac{1}{2\cdot 2} \ln\left|\frac{u-2}{u+2}\right| + C \\
&= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C
\end{aligned}$$

7.5 Strategy for integration

There is no step-by-step procedure for taking any integral. In fact, anti-derivatives of most of elementary functions are not elementary functions.

Anyway, try the following steps:

1. Simplify the integrand, if possible
2. Look for a substitution
3. Classify the integrand in terms of sections 7.2 - 7.4
4. Try to integrate by parts
5. Try all the above again

7.8 Improper integrals

Definition

If $\int_a^t f(x)\,dx$ or $\int_t^b f(x)\,dx$ exist $\forall t \geq 0$, then

$$\begin{aligned} \int_a^{\infty} f(x)\,dx &= \lim_{t\to\infty} \int_a^t f(x)\,dx \\ \int_{-\infty}^{b} f(x)\,dx &= \lim_{t\to-\infty} \int_t^b f(x)\,dx \end{aligned}$$

provided that the limits exist, in which case the corresponding integral is called **convergent** and **divergent** otherwise.
If both limits exist, we define (a can be any)

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

Those are improper integrals of **Type 1**.

Example

Evaluate $\int_{-\infty}^{0} xe^x \, dx$. We integrate by parts, $u = x$, $dv = e^x \, dx$, so $du = dx$, $v = e^x$

$$\begin{aligned}\int_{-\infty}^{0} xe^x \, dx &= \lim_{t \to -\infty} \int_t^0 xe^x \, dx = \lim_{t \to -\infty} xe^x]_t^0 - \int_t^0 e^x \, dx \\ &= \lim_{t \to -\infty} \left(-te^t - 1 + e^t\right) = -0 - 1 + 0 = -1\end{aligned}$$

Example

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$. With $a = 0$ we have

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \int_{-\infty}^{0} \frac{1}{1+x^2} \, dx + \int_{0}^{\infty} \frac{1}{1+x^2} \, dx$$

We work out the integrals separately.

$$\begin{aligned}\int_0^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \tan^{-1} x\Big]_0^t \\ &= \lim_{t\to\infty}(\tan^{-1} t - \tan^{-1} 0) = \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \tan^{-1} x\Big]_t^0 \\ &= \lim_{t\to-\infty}(\tan^{-1} 0 - \tan^{-1} t) \\ &= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}\end{aligned}$$

Hence, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Example

For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Big]_1^t \qquad (p \neq 1)$$

$$= \lim_{t\to\infty} \frac{1}{1-p}\left[\frac{1}{t^{p-1}} - 1\right]$$

Hence, $\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p > 1 \\ \lim_{t\to\infty} t^{1-p} \to \infty, & \text{if } p < 1 \end{cases}$

For $p = 1$ it holds: $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t = \lim_{t\to\infty} \ln t = \infty.$

Therefore, the integral $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent for $p > 1$.

Type 2: Discontinuous Integrals

Definition

If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

provided that the limits exist, in which case the corresponding integral is called **convergent** and **divergent** otherwise.
If f has discontinuity at c, $a < c < b$, and **both** limits exist, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} 2\sqrt{x-2}\Big]_t^5$$
$$= \lim_{t\to 2^+} 2(\sqrt{3}-\sqrt{t-2}) = 2\sqrt{3}$$

Example

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+}\int_t^1 \ln x\,dx \qquad (u=\ln x,\; dv=dx)$$
$$= \lim_{t\to 0^+} x\ln x\Big]_t^1 - \int_t^1 dx$$
$$= \lim_{t\to 0^+}(1\ln 1 - t\ln t-(1-t)) = \lim_{t\to 0^+}(-t\ln t - 1 + t)$$
$$= \lim_{t\to 0^+}\frac{\ln t}{1/t} - 1 = \lim_{t\to 0^+}\frac{1/t}{-1/t^2} - 1 = \lim_{t\to 0^+}(-t) - 1 = -1$$

Example

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

However,

$$\int_0^1 \frac{dx}{x-1} = \lim_{t \to 1^-} \int_0^t \frac{dx}{x-1} = \lim_{t \to 1^-} \ln|x-1| = -\infty$$

So, we do not need to evaluate the second integral and the original one is divergent.

Erroneous calculation

$$\int_0^1 \frac{dx}{x-1} = \ln|x-1|]_1^2 = \ln 2 - \ln 1 = \ln 2$$

because the integral is improper and **must** be split into two integrals.

Example

$$\begin{aligned}
\int_{-1}^{0} \frac{e^{1/x}}{x^3}\,dx &= \lim_{t\to 0^-} \int_{-1}^{t} \frac{e^{1/x}}{x^3}\,dx \\
&= -\lim_{t\to 0^-} \int_{-1}^{t} \frac{e^{1/x}}{x}\,d(1/x) \\
&= \lim_{t\to\infty} \int_{-t}^{-1} ue^{u}\,d(u) \qquad (u = 1/x) \\
&= \lim_{t\to\infty} \left(ue^{u}\,|_{-t}^{-1} - \int_{-t}^{-1} e^{u}\,du \right) \\
&= \lim_{t\to\infty} e^{u}(u-1)|_{-t}^{-1} \\
&= -\frac{2}{e} + \lim_{t\to\infty} (t+1)e^{-t} \\
&= -\frac{2}{e}
\end{aligned}$$

Comparison Theorem

Theorem

Suppose f and g are continuous and $f(x) \geq g(x) \geq 0$ for $x \geq a$.

- *If $\int_a^\infty f(x)\,dx$ is convergent, then so it is $\int_a^\infty g(x)\,dx$*
- *If $\int_a^\infty g(x)\,dx$ is divergent, then so it is $\int_a^\infty f(x)\,dx$*

Example

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

The first integral is ordinary. The second one is convergent since $e^{-x} \geq e^{-x^2}$ for $x \geq 1$ and $\int_1^\infty e^{-x}\,dx$ is convergent.

Example

The integral $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent because

$$\frac{1+e^{-x}}{x} \geq \frac{1}{x} \qquad \text{for} \quad x \geq 1$$

and $\int_1^\infty \frac{1}{x}\,dx$ is divergent.