Outline

6.1 Definition

Definition

A function f is called **one-to-one** if

$$f(x_1) \neq f(x_2) \qquad \text{whenever} \qquad x_1 \neq x_2$$

A function is one-to-one iff no horizontal line intersects its graph more than twice.

Definition

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \qquad \text{iff} \qquad f(x) = y$$

Note that:

$$\forall x \in A : f^{-1}(f(x)) = x$$

$$\forall x \in B : f(f^{-1}(x)) = x$$

The graph of $f^{-1}(x)$ is obtained by reflecting the one of f about the line $y = x$.

To find inverse of a one-to-one function f:

1. Write $y = f(x)$
2. Interchange x and y
3. Solve the obtained equation for y

Example

Find the inverse of $f(x) = x^3 + 2$.

1. Write $y = f(x)$:
$$y = x^3 + 2$$
2. Interchange x and y:
$$x = y^3 + 2$$
3. Solve the obtained equation for y:
$$y^3 = x - 2 \qquad \Longrightarrow \qquad y = \sqrt[3]{x - 2}$$
Finally,
$$f^{-1}(x) = \sqrt[3]{x - 2}$$

The Calculus of Inverse Functions

Theorem

If f is a one-to one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

Informally, for $f(b) = a$ (thus, $f^{-1}(a) = b$) one has

$$(f^{-1})'(a) = \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x / \Delta y} = \frac{1}{f'(b)}$$

Theorem

If f is one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$\boxed{(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}}$$

Proof.

Let $f(y) = x$ and $f(b) = a$. Hence, $f^{-1}(x) = y$ and $f^{-1}(a) = b$.

Since f is differentiable, f is continuous, so f^{-1} is continuous.

Therefore, $f^{-1}(x) \to f^{-1}(a)$ (i.e. $y \to b$) as $x \to a$.

$$\begin{aligned}
(f^{-1})'(a) &= \lim_{x\to a} \frac{f^{-1}(x) - f^{-1}(a)}{x - a} = \lim_{y\to b} \frac{y - b}{f(y) - f(b)} \\
&= \lim_{y\to b} \frac{1}{\frac{f(y)-f(b)}{y-b}} = \frac{1}{\lim_{y\to b} \frac{f(y)-f(b)}{y-b}} \\
&= \frac{1}{f'(b)} = \frac{1}{f'(f^{-1}(a))}
\end{aligned}$$

$\square$

Example

For $y = x^2$ and $0 \leq x \leq 2$ one has

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(1)} = \frac{1}{2}$$

Example

For $y = 2x + \cos x$ the function y is increasing, hence one-to-one.

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{2 - \sin 0} = \frac{1}{2}$$

6.2 Exponential Functions and Their Derivatives

Definition

Exponential function is a function of the form

$$f(x) = a^x \qquad \text{for} \quad a > 0$$

If $x = n$ a positive integer,

$$a^n = a \cdot a \cdots a \qquad n \ \text{terms}$$

If $x = 0$ then $a^0 = 1$, and if $x = -n$ then

$$a^{-n} = \frac{1}{a^n}$$

If $x = p/q$ is rational then

$$a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$$

In general define

$$\boxed{a^x = \lim_{r \to x} a^r \qquad r \ \text{rational}}$$

Theorem

If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is continuous with domain $I\!R$ and range $(0, \infty)$. For $0 < a < 1$, a^x is decreasing and for $1 < a$, a^x is increasing. Furthermore, $\forall a, b > 0$ and $x, y \in I\!R$,

$$a^{x+y} = a^x a^y \qquad a^{x-y} = \frac{a^x}{a^y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

Let $f(x) = a^x$. Then

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x (a^h - 1)}{h} \\
&= a^x \lim_{h \to 0} \frac{a^h - 1}{h} \\
&= f'(0) a^x
\end{aligned}
$$

Definition

The Number e is defined by equation

$$
\boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1}
$$

One has then $(e^x)' = e^x$.

Example

$$\lim_{x\to\infty} \frac{e^{2x}}{e^{2x}+1} = 1$$

Example

Sketch the graph of $f(x) = e^{1/x}$.

Example

$$\int x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C$$

6.3 Logarithmic Functions

Definition

Logarithmic function is the inverse of exponential one.

Example

$\log_3 81 = 4$ because $3^4 = 81$.
$\log_2 32 = 5$ because $2^5 = 32$.

Note that

$$\log_a x = y \qquad \text{iff} \qquad a^y = x$$

It follows:

$$\begin{aligned} \log_a(a^x) &= x \qquad \forall x \in I\!R \\ a^{\log_a x} &= x \qquad \forall x > 0 \end{aligned}$$

Properties of logarithmic function

Theorem

If $x, y > 0$ and r is a real number then:

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r \log_a x$

If $a > 1$ then

$$\lim_{x\to\infty} \log_a x = +\infty \qquad \text{and} \qquad \lim_{x\to 0^+} \log_a x = -\infty$$

If $a < 1$ then

$$\lim_{x\to\infty} \log_a x = -\infty \qquad \text{and} \qquad \lim_{x\to 0^+} \log_a x = +\infty$$

Natural logarithm:

$$\log_e x = \ln x$$

Hence,

$$\ln(e^x) = x \qquad \forall x \in I\!R$$
$$e^{\ln x} = x \qquad \forall x > 0$$

Theorem

Change of base formula:

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

Proof.

If $y = \log_a x$ then $a^y = x$. Taking the ln of both parts, we get

$$y \ln a = \ln x \qquad \text{i.e.} \qquad y = \frac{\ln x}{\ln a}$$

□

Corollary

$$\boxed{\log_b a = \frac{1}{\log_a b}} \qquad \text{for } a, b > 0$$

6.4 Derivatives of Logarithmic Functions

Theorem

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Proof.

If $y = \ln x$ then $e^y = x$. Differentiating implicitly by x we get

$$e^y \frac{dy}{dx} = 1$$

Hence,

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

□

In general we get

$$\boxed{\frac{d}{dx}[\ln(g(x)] = \frac{g'(x)}{g(x)}}$$

Example

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

Example

$$\begin{aligned}\frac{d}{dx}\sqrt{\ln x} &= \frac{1}{2}(\ln x)^{-\frac{1}{2}}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x} \\ &= \frac{1}{2x\sqrt{\ln x}}\end{aligned}$$

Example

$$\begin{aligned} \frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} &= \frac{1}{\frac{x+1}{\sqrt{x-2}}} \cdot \frac{d}{dx}\left[\frac{x+1}{\sqrt{x-2}}\right] \\ &= \frac{\sqrt{x-2}}{x+1} \cdot \frac{\sqrt{x-2} - (x+1)\frac{1}{2}(x-2)^{-\frac{1}{2}}}{x-2} \\ &= \frac{x-2-\frac{1}{2}(x+1)}{(x+1)(x-2)} = \frac{x-5}{2(x+1)(x-2)} \end{aligned}$$

Another solution:

$$\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left[\ln(x+1) - \frac{1}{2}\ln(x-2)\right] = \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

Example

Find $\frac{d}{dx}\ln|x|$

Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

We get

$$f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x} & \text{if } x < 0 \end{cases}$$

Therefore,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

and

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

Example

Evaluate $\int \frac{x}{x^2+1}dx$. Use substitution $u = x^2 + 1$, so $du = 2x\,dx$:

$$\begin{aligned}\int \frac{x}{x^2+1}dx &= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}\ln|x^2+1| + C = \frac{1}{2}\ln(x^2+1) + C\end{aligned}$$

Example

With $u = \ln x$ one has

$$\int_1^e \frac{\ln x}{x}dx = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}$$

Example

Calculate $\int \tan x \, dx$. Note that

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

Substituting $u = \cos x$ (hence, $du = -\sin x \, dx$) we get

$$\begin{aligned}
\int \tan x \, dx &= \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} \\
&= -\ln|u| + C = -\ln|\cos x| + C \\
&= \ln \frac{1}{|\cos x|} + C = \ln|\sec x| + C
\end{aligned}$$

General Logarithmic and Exponential Functions

Since $\log_a x = \frac{\ln x}{\ln a}$ we get

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$$

Theorem

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a} \qquad \boxed{\int a^x \, dx = \frac{a^x}{\ln a} + C \quad a \neq 1}$$

Proof.

We use the fact that $e^{\ln a} = a$:

$$\begin{aligned} \frac{d}{dx}(a^x) &= \frac{d}{dx}(e^{\ln a})^x = \frac{d}{dx} e^{(\ln a)x} = (e^{\ln a})^x \frac{d}{dx}((\ln a)x) \\ &= (e^{\ln a})^x (\ln a) = a^x \ln a \end{aligned}$$

Example

$$\frac{d}{dx}\log_{10}(2+\sin x) = \frac{1}{(2+\sin x)\ln 10}\frac{d}{dx}(2+\sin x) = \frac{\cos x}{(2+\sin x)\ln 10}$$

Example

$$\frac{d}{dx}(10^{x^2}) = 10^{x^2}(\ln 10)\frac{d}{dx}(x^2) = (2\ln 10)x10^{x^2}$$

Example

$$\int_2^5 2^x dx = \frac{2^x}{\ln 2}\Big]_0^5 = \frac{2^5}{\ln 2} - \frac{2^0}{\ln 2} = \frac{31}{\ln 2}$$

Logarithmic Differentiation

Example

Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$ Take ln of both sides

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

and differentiate it implicitly:

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for y' results

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Finally, we get

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Steps in logarithmic differentiation:

1. Take ln of both sides of an equation $y = f(x)$ and simplify it
2. Differentiate implicitly with respect to x
3. Solve the resulting equation for y'

The Number e as a Limit

For $f(x) = \ln x$ we have

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{x\to 0} \frac{f(1+x)-f(1)}{x} \\
&= \lim_{x\to 0} \frac{\ln(1+x)-\ln 1}{x} = \lim_{x\to 0} \frac{1}{x}\ln(1+x) \\
&= \lim_{x\to 0} \ln(1+x)^{1/x} = 1
\end{aligned}$$

From here it follows that

$$\boxed{e = \lim_{x\to 0}(1+x)^{1/x}}$$

If we put $n = 1/x$, then $n \to \infty$ as $x \to 0$, hence

$$\boxed{e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}}$$

6.6 Inverse Trigonometric Functions

$y = \sin^{-1} x$ or $y = \arcsin x$ with $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$y = \cos^{-1} x$ or $y = \arccos x$ with $0 \leq y \leq \pi$

$y = \tan^{-1} x$ or $y = \arctan x$ with $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$y = \cot^{-1} x$ with $0 \leq y \leq \pi$

$y = \csc^{-1} x$ with $y \in (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

$y = \sec^{-1} x$ with $y \in (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Proof.

If $y = \sin^{-1} x$ then $\sin y = x$. Differentiating implicitly we get $(\cos y) \cdot y' = 1$, So

$$y' = \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

Example

Differentiate $y = \dfrac{1}{\sin^{-1} x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\sin^{-1} x)^{-1} = -(\sin^{-1} x)^{-2}\frac{d}{dx}(\sin^{-1} x)\\ &= -\frac{1}{(\sin^{-1} x)^2\sqrt{1-x^2}}\end{aligned}$$

The formulas in the frame box on the previous page can be rewritten as

$$\boxed{\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C}$$

$$\boxed{\int \frac{1}{x^2+1}dx = \tan^{-1} x + C}$$

Example

Evaluate $\int \frac{dx}{x^2 + a^2}$, where $a = \text{const}$ and $a \neq 0$.

$$\int \frac{dx}{x^2 + a^2} = \int \frac{dx}{a^2 \left(\frac{x^2}{a^2} + 1 \right)} = \frac{1}{a^2} \int \frac{dx}{\left(\frac{x}{a} \right)^2 + 1}$$

Substitute $u = x/a$. Then $du = dx/a$ and $dx = a\,du$, so

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a^2} \int \frac{a\,du}{u^2 + 1} = \frac{1}{a} \int \frac{du}{u^2 + 1} = \frac{1}{a} \tan^{-1} u + C$$

This implies

$$\boxed{\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C}$$

6.7 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \cdot \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \cdot \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Proof.

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

□

Inverse Hyperbolic Functions

$y = \sinh^{-1} x$	$\Longleftrightarrow$	$\sinh y = x$
$y = \cosh^{-1} x$	$\Longleftrightarrow$	$\cosh y = x$
$y = \tanh^{-1} x$	$\Longleftrightarrow$	$\tanh y = x$

One has

$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$	$x \in I\!R$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$	$x \geq 1$
$\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1 + x}{1 - x}\right)$	$-1 < x < 1$

Derivatives of Inverse Hyperbolic Functions

$$
\begin{aligned}
\frac{d}{dx}(\sinh^{-1} x) &= \frac{1}{\sqrt{x^2+1}} & \qquad \frac{d}{dx}(\operatorname{csch}^{-1} x) &= -\frac{1}{|x|\sqrt{x^2+1}} \\
\frac{d}{dx}(\cosh^{-1} x) &= \frac{1}{\sqrt{x^2-1}} & \qquad \frac{d}{dx}(\operatorname{sech}^{-1} x) &= -\frac{1}{x\sqrt{1-x^2}} \\
\frac{d}{dx}(\tanh^{-1} x) &= \frac{1}{1-x^2} & \qquad \frac{d}{dx}(\coth^{-1} x) &= \frac{1}{1-x^2}
\end{aligned}
$$

Proof.

Let $y = \sinh^{-1} x$, then $\sinh y = x$ and $(\cosh x)\frac{dy}{dx} = 1$. Hence,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

$\square$

Example

$$
\begin{aligned}
\frac{d}{dx}[\tanh^{-1}(\sin x)] &= \frac{1}{1-\sin^2 x}(\sin x)' \\
&= \frac{1}{1-\sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x
\end{aligned}
$$

Example

$$
\begin{aligned}
\int_0^1 \frac{dx}{\sqrt{1+x^2}} &= \sinh^{-1} x\Big]_0^1 \\
&= \sinh^{-1} 1 \\
&= \ln(1+\sqrt{2})
\end{aligned}
$$

6.8 Indeterminate Forms and l'Hospital's Rule

Theorem

Suppose $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ on an open interval containing a. Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(indeterminate forms of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$). Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the R.H.S. exists (or is ∞ or $-\infty$).

Example

Find $\lim_{x\to 1} \dfrac{\ln x}{x-1}$. It is indeterminate form of type $\frac{0}{0}$.

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1$$

Example

Find $\lim_{x\to\infty} \dfrac{e^x}{x^2}$. We got an indeterminate form of type $\frac{\infty}{\infty}$.

We apply l'Hospital's rule twice:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

Example

Find $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$. l'Hospital's rule applies:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0$$

Example

Find $\lim_{x\to 0} \frac{\tan x - x}{x^3}$. This indeterminate form of type $\frac{0}{0}$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \\
&= \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x} \\
&= \frac{1}{3} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{3} \lim_{x\to 0} \frac{\cos x}{1} = \frac{1}{3}
\end{aligned}$$

Indeterminate Products

Assume $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ and we need to compute $\lim_{x \to a} f(x) \cdot g(x)$. This is indeterminate form of type $0 \cdot \infty$.

We turn it to indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by rewriting it as

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

Example

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

Indeterminate Differences

Assume $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ and we need to compute $\lim_{x\to a}(f(x) - g(x))$. This is indeterminate form of type $\infty - \infty$.

We turn it to indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by using a common denominator, or rationalization, or factoring out a common factor.

Example

$$\begin{aligned}
\lim_{x\to(\pi/2)^-}(\sec x - \tan x) &= \lim_{x\to(\pi/2)^-}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) \\
&= \lim_{x\to(\pi/2)^-}\frac{1 - \sin x}{\cos x} \\
&= \lim_{x\to(\pi/2)^-}\frac{-\cos x}{-\sin x} = 0
\end{aligned}$$

Indeterminate Powers

The following situations can occur while computing the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$: type 0^0
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$: type ∞^0
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$: type 1^∞

We turn it into indeterminate form of type $0 \cdot \infty$ by taking the logarithm $\ln\left([f(x)]^{g(x)}\right) = g(x)\ln f(x)$ and using the exponentiation

$$\lim_{x\to a}[f(x)]^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)}$$

Example

Find $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$. For $y=(1+\sin 4x)^{\cot x}$ we have $\ln y = \cot x \ln(1+\sin 4x)$, so

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = 4$$

Hence, $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = e^4$.

Example

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1$$

Cauchy's Mean Value Theorem

Theorem

Assume $f(x)$ *and* $g(x)$ *are continuous on* $[a, b]$ *and differentiable on* (a, b)*, and* $g'(x) \neq 0$ *for all* $x \in (a, b)$*. Then there is a number* $c \in (a, b)$ *such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Note that for $g(x) = x$ (so $g'(x) = 1$) the theorem becomes the Mean Value Theorem.

In general, the proof can be deduced along the lines with the proof of Rolle's theorem by applying the argument to the function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)]$$

Proof of L'Hospital's Rule

Assuming $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, denote

$$L = \lim_{x\to a} = \frac{f'(x)}{g'(x)}$$

We show that $\lim_{x\to a}[f(x)/g(x)] = L$. For that define

$$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases} \qquad G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$$

$F(x)$ and $G(x)$ are continuous on I. Let $x \in I$ and $x > a$. Then $F(x)$ and $G(x)$ are continuous on $[a, x]$ and differentiable on (a, x) and $G'(x) \neq 0$ there. By Cauchy's theorem,

$$\frac{F'(y)}{G'(y)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)} \qquad \text{for some } y \in (a, x)$$

If we let $x \to a^+$, then $y \to a^+$, so

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{F(x)}{G(x)} = \lim_{x\to a^+} \frac{F'(x)}{G'(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$$

Similar argument also works for $x \to a^-$.

If a is infinite, we let $t = 1/x$. Then $t \to 0^+$ as $x \to \infty$, so

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{t\to 0^+} \frac{f(1/t)}{g(1/t)} \\
&= \lim_{t\to 0^+} \frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} \\
&= \lim_{t\to 0^+} \frac{f'(1/t)}{g'(1/t)} \\
&= \lim_{x\to\infty} \frac{f'(x)}{g'(x)}
\end{aligned}$$