Outline

11.1 Sequences

A sequence is a list of numbers written in a definite order

$$a_1, a_2, \ldots, a_n, \ldots$$

The sequence $\{a_1, a_2, \ldots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Example

$$\left\{\frac{n}{n+1}\right\}_{n=2}^{\infty} \qquad \{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

Example

The general term of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \ldots\right\}$$

is obviously

$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$

Definition

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\epsilon > 0$ there is N such that

$$|a_n - L| < \epsilon \quad \text{whenever} \quad n > N$$

If the lim exists the sequence is called **convergent** and **divergent** otherwise.

Theorem

If $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$ *then* $\lim_{n\to\infty} a_n = L$.

Example

The sequence $1/n^r$ is convergent for $r \geq 0$ and divergent otherwise.

Definition

$\lim_{n\to\infty} a_n = \infty$ means that for every positive M there is an N such that

$$a_n > M \quad \text{whenever} \quad n > N$$

We say that $\{a_n\}$ diverges to infinity.

If $\{a_n\}$ and $\{b_n\}$ are convergent and c is a constant then

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \,/\, \lim_{n\to\infty} b_n \qquad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^c = \left[\lim_{n\to\infty} a_n\right]^c \qquad \text{if } c > 0 \text{ and } a_n > 0$$

The following theorems can be adopted from functions to sequences

Theorem

If $a_n \leq b_n \leq c_n$ *and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ *then* $\lim_{n\to\infty} b_n = L$.

Theorem

If $\lim_{n\to\infty} |a_n| = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Example

Find $\lim_{n\to\infty} \frac{n}{n+1}$. One has

$$\begin{aligned} \lim_{n\to\infty} \frac{n}{n+1} &= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+\lim_{n\to\infty} \frac{1}{n}} \\ &= \frac{1}{1+0} = 1 \end{aligned}$$

Example

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{n^2}+\frac{10}{n}}} = \infty$$

Example

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{n\to\infty} \frac{1/x}{1} = 0 \quad (\mathrm{l'Hospital'sRule})$$

Example

The sequence $a_n = (-1)^n$ is divergent.

Example

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0 \quad \text{since} \quad \lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Theorem

If $\lim_{n\to\infty} a_n = L$ *and* f *is continuous at* L *then*

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Example

$$\lim_{n\to\infty} \sin(\pi/n) = \sin\left(\lim_{n\to\infty}(\pi/n)\right) = \sin 0 = 0$$

Example

For the sequence $a_n = n!/n^n$ we cannot apply the l'Hospital's rule. However,

$$0 < a_n = \frac{1}{n}\left(\frac{2\cdot 3\cdot\cdots\cdot n}{n\cdot n\cdot\cdots\cdot n}\right) < \frac{1}{n}$$

Hence, $\lim_{n\to\infty} a_n = 0$ by the Squeeze Theorem.

Properties of exponential functions imply

$$\lim_{n\to\infty} r^n = \begin{cases} \infty, & \text{if } r > 1 \\ 1, & \text{if } r = 1 \\ 0, & \text{if } 0 \le r < 1 \end{cases}$$

Hence, $\{r^n\}$ is convergent for $-1 < r \le 1$ and divergent for all other values.

Definition

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is called **monotonic** if it is either increasing or decreasing.

Example

The sequence $\left\{\frac{3}{n+5}\right\}$ is decreasing because

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

Example

The sequence $a_n = \frac{n}{n^2+1}$ is decreasing because $a_{n+1} < a_n$ is equivalent to

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \quad \Longleftrightarrow \quad 1 < n^2 + n$$

Alternatively, the sequence is decreasing because the function $f(x) = \frac{x}{x^2+1}$ is decreasing:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0$$

Definition

A sequence $\{a_n\}$ is called **bounded above** if there is M such that

$$a_n \leq M \quad \text{for all} \quad n \geq 1$$

and **bounded below** if there is m such that

$$m \leq a_n \quad \text{for all} \quad n \geq 1$$

If $\{a_n\}$ is bounded above and below it is called **bounded.**

Example

The sequence $\{\frac{n}{n+1}\}$ is bounded since its general term a_n satisfies $0 < a_n < 1$. In this case 1 is its least upper bound.

Theorem

Every bounded monotonic sequence is convergent.

Proof.

If $\{a_n\}$ is increasing bounded, by the Completeness Axiom it has a least upper bound L. Since $L - \epsilon$ is not an upper bound and $\{a_n\}$ is increasing then $L - \epsilon < a_n \leq L$ whenever $n > N$ for some N. Thus, $\lim_{n\to\infty} a_n = L$.
The proof for decreasing sequences is similar. □

Example

The sequence $a_n = 1 - \frac{1}{n}$ is increasing and bounded by 1. Hence, it is convergent. Moreover, $\lim_{n\to\infty} a_n = 1$

11.2 Series

For a sequence $\{a_n\}$ the following sum is called **series**:

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n + \cdots$$

Definition

Given a series $\sum_{i=1}^{\infty} a_i$, let s_n denote its partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Is the sequence s_n is convergent to a real number s then the series $\sum a_i$ is called **convergent** and s is called its **sum.** Otherwise, the series is called **divergent.**

Example

Geometric series for $a \neq 0$:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{i=1}^{\infty} ar^{i-1}$$

If $r = 1$ the series is obviously divergent. For $r \neq 1$ we have:

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n \end{aligned}$$

So, $s_n - rs_n = a - ar^n$ and $s_n = \frac{a(1-r^n)}{1-r}$. Therefore, the geometric series is convergent if $|r| < 1$ and

$$\sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r} \qquad |r| < 1$$

Example

Is the series $\sum_{n=1}^{\infty} 2^{2n}\, 3^{1-n}$ convergent or divergent?
We rewrite the general term in the form ar^{n-1}:

$$2^{2n}\, 3^{1-n} = \left(2^2\right)^n\, 3^{-(n-1)} = \frac{4^n}{3^{n-1}} = 4\left(\frac{4}{3}\right)^{n-1}$$

So, $a = 4$ and $r = 4/3$. Since $r > 1$ the series is divergent.

Example

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum. One has

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) = 1 - \frac{1}{n+1}$$

Hence, the series converges to 1 because

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

Example

Show that the **Harmonic series** is divergent

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

We have

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2} \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}
\end{aligned}$$

Similarly, $s_{2^n} > 1 + \frac{n}{2}$, so the series is divergent.

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent then* $\lim_{n\to\infty} a_n = 0$*.*

Proof.

For $s_n = a_1 + a_2 + \cdots + a_n$ we have $a_n = s_n - s_{n-1}$. Since $\sum a_n$ is convergent, s_n is convergent to some number s. One has

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

$\square$

Corollary

Test for divergence: *if* $\lim_{n\to\infty} a_n \neq 0$ *or does not exist, then the series is divergent.*

Example

The series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ is divergent because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0$$

Note that the Test for convergence only works in one direction and its converse is not true, in general.

Example

The Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ passes the Test for convergence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

However, the series is divergent. On the other hand, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ also passes the Test and is convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

Theorem

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series ca_n (c is a constant) and $\sum(a_n \pm b_n)$. Moreover

- $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
- $\displaystyle\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

Example

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right)$

For the sum of geometric series, $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. Furthermore,

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 3$$

So, the total sum is $3 + 1 = 4$.

11.3 The Integral Test and Estimates of Sums

Theorem

Suppose f is continuous, positive, decreasing on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent iff the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.

Proof.

The proof follows immediately from the inequalities

$$\int_1^n f(x)\,dx + a_n \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx$$

$\square$

Note that for convergent series $\sum_{n=1}^{\infty} f(n) \neq \int_1^{\infty} f(x)\,dx$, in general. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{whereas} \quad \int_1^{\infty} \frac{1}{x^2}\,dx = 1$$

Example

$\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent because

$$\begin{aligned}\int_1^{\infty} \frac{\ln x}{x}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx = \lim_{n\to\infty} \left.\frac{(\ln x)^2}{2}\right]_1^t \\ &= \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty\end{aligned}$$

Example

For what values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?
The series is obviously divergent for $p \leq 0$. If $p > 0$ then $f(x) = 1/x$ is continuous, positive, and decreasing. As we already know

$$\int_1^{\infty} \frac{1}{x^p}\,dx \quad \text{is convergent only for } p > 1$$

So, the series is convergent for $p > 1$, otherwise divergent.

Estimating the Sums of Series

Assume $\sum a_n = \sum f(n)$ is convergent series and we want to find an approximation for its sum s. For this we estimate the **remainder**

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

By using a similar approach as in the Integral Test Theorem,

$$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_{n}^{\infty} f(x)\, dx$$

Example

Estimate the error of approximation of $\sum(1/n^3)$ with s_{10}.
With $f(x) = 1/x^3$ we get

$$\int_n^{\infty} \frac{dx}{x^3} = \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

For the remainder it holds

$$R_{10} \leq \int_n^{\infty} \frac{dx}{x^3} = \frac{1}{2 \cdot 10^2} = 0.005$$

How many terms of the sum should we take to reach the accuracy 0.0005?

The inequality $R_n \leq 0.0005$ is equivalent to $\frac{1}{2n^2} \leq 0.0005$ from where $n \geq 32$ follows.

A better approximation to the sum $\sum a_n$ follows from $s_n + R_n = s$ and the estimates of R_n from above:

$$s_n + \int_{n+1}^{\infty} f(x)\, dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\, dx$$

Example

To estimate $\sum_{n=1}^{\infty}(1/n^3)$ we apply the above formula with $n = 10$:

$$s_{10} + \int_{11}^{\infty} \frac{dx}{x^3} \leq s \leq s_{10} + \int_{10}^{\infty} \frac{dx}{x^3}$$

from where we get

$$s_{10} + \frac{1}{2 \cdot 11^2} \leq s \leq s_{10} + \frac{1}{2 \cdot 10^2}$$

Using $s_{10} \approx 1.197532$ we obtain

$$1.201664 \leq s \leq 1.202532$$

Hence, the sum is approx. 1.2021 with error < 0.0005.

11.4 The Comparison Tests

Theorem

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- *If $a_n \leq b_n$ for all n and $\sum b_n$ is convergent then $\sum a_n$ is convergent.*
- *If $a_n \geq b_n$ for all n and $\sum b_n$ is divergent then $\sum a_n$ is divergent.*

Proof.

$$\text{Denote} \qquad s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum^{\infty} b_{i=1}$$

- Since s_n and t_n are increasing, $s_n \leq t_n$ and $s_n \leq t$. By the Monotonic Sequence Theorem $\sum a_n$ is convergent.
- Since $a_n \geq b_n$, $s_n \geq t_n$. Thus $s_n \to \infty$.

$\square$

Most of the time the power series $\sum 1/n^p$ (convergency for $p > 1$ only) or geometric series are used for comparison.

Example

Investigate the series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ for convergency. One has

$$\frac{5}{2n^2+4n+3} \leq \frac{5}{2n^2} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} \leq \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ is convergent, so is the series in question.

Example

Investigate the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergency. One has

$$\frac{\ln n}{n} \geq \frac{1}{n} \qquad \text{for } n \neq 3$$

Since the Harmonic series $\sum \frac{1}{n}$ is divergent, so is the one in question.

Theorem

Suppose $\sum a_n$ and $\sum b_N$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

for some finite $c > 0$ then either both series converge of both diverge.

Proof.

Since a_n/b_n converges to c, for large $n > N$ and some m, M with $m < c < M$ we have

$$m < \frac{a_n}{b_n} < M \qquad \Longleftrightarrow \qquad mb_n < a_n < Mb_n \qquad \text{for } n > N$$

If $\sum b_n$ converges, so does $\sum Mb_n$, hence $\sum a_n$ converges by the Comparison Test.
Similarly, if $\sum b_n$ diverges, so does $\sum mb_n$, hence $\sum a_n$ diverges. $\square$

Example

Investigate the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ for convergency.

The dominant terms of the numerator and denominator are n^2 and n^5, respectively. This suggests taking

$$a_n = \frac{2n^2+3n}{\sqrt{5+n^5}} \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{1/2}}{2\sqrt{5+n^5}} \\ &= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}} = \frac{2+0}{2\sqrt{0+1}} = 1 \end{aligned}$$

Since $\displaystyle\sum \frac{1}{n^{1/2}}$ is divergent, so is the series in question.

Estimating Sums

If $\sum a_n$ and $\sum b_n$ pass the comparison test, $a_n \leq b_n$ and $\sum b_n$ is convergent then $R_n \leq T_n$ where

$$\begin{aligned} R_n &= s - s_n = a_{n+1} + a_{n+2} + \cdots \\ T_n &= t - t_n = b_{n+1} + b_{n+2} + \cdots \end{aligned}$$

If $\sum b_n$ is geometric series, it is easy to estimate the R_n.

Example

For the series $\sum 1/(n^3+1)$ and $\sum 1/n^3$ we have $1/(n^3+1) < 1/n^3$. Earlier we showed $T_n \leq \int_n^\infty \frac{dx}{x^3} = \frac{1}{2n^2}$. Hence for the remainder term of the first series one has

$$R_n \leq T_n \leq \frac{1}{2n}$$

For $n = 100$, $R_n \leq 0.0005$ and $\sum_{n=1}^{100} 1/(n^3+1) \approx 0.6864538$ with accuracy 0.0005.

11.5 Alternating Series

An alternating series is a series whose terms are alternately positive or negative. Example:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

Theorem

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ *satisfies*

- $b_{n+1} \leq b_n$, *for all* n
- $\lim_{n \to \infty} b_n = 0$

then the series is convergent.

Proof.

We first consider the even partial sums

$$\begin{aligned} s_2 &= b_1 - b_2 \geq 0 \\ s_4 &= s_2 + (b_3 - b_4) \geq s_2 \\ s_{2n} &= s_{2n-2} + (b_{2n-1} - b_{2n}) \geq s_{2n-2} \end{aligned}$$

On the other hand,

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \leq b_1$$

Since s_{2n} is increasing and bounded it is convergent: $\lim_{n\to\infty} s_{2n} = s$ for some s. For the odd partial sums we have:

$$\begin{aligned} \lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\ &= s + 0 = s \end{aligned}$$

Hence, for any partial sum we have $\lim_{n\to\infty} s_n = s$. $\square$

Example

The alternating Harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent because

- $b_{n+1} < b_n$ is equivalent to $\frac{1}{n+1} < \frac{1}{n}$
- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} 1/n = 0$

Example

For the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ we have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0$$

Hence, the previous theorem is not applicable. However, since the following limit does not exist, the series is divergent:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

Estimating Sums

Theorem

If $s = \sum(-1)^{n-1} b_n$ is the sum of alternating series such that

- $b_{n+1} \leq b_n$
- $\lim_{n\to\infty} b_n = 0$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Indeed, since s_n is larger than all even partial sums and smaller than all odd ones,

$$|s - s_n| \leq |s_{n+1} - s_n| = b_{n+1} \qquad \square$$

Example

Compute $\sum_{n=0}^{\infty}(-1)^n/n!$ correct to 3 decimal places. The conditions of the theorem are satisfied. Since $b_7 \leq 0.0002$,

$$|s - s_6| \leq b_7 \leq 0.0002$$

So, by summing up the first 6 terms we get $s \approx 0.368056$.

11.6 Absolute Convergence. Ratio and Root Tests

Definition

A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
If the series is convergent but not absolutely convergent, it is called **conditionally convergent**.

Example

The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent, since the power series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Example

We know that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent. Since the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the series is conditionally convergent.

Theorem

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Proof.

Note that $0 \leq a_n + |a_n| \leq 2|a_n|$. The series $\sum 2|a_n|$ is convergent and so is $\sum(a_n + |a_n|)$ by the Comparison Test. So,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

Hence, the series $\sum_{n=1}^{\infty} a_n$ is convergent. $\square$

Example

Show that the series $\sum_{n=1}^{\infty} \cos n/n^2$ is convergent.

The series has positive and negative terms but is not alternating. We apply the above theorem:

$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The Ratio Test

Theorem

- *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent (hence, simply convergent).*
- *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$*, then the series is divergent.*

Proof.

For the first statement, choose r such that $L < r < 1$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| < r \quad \Leftrightarrow \quad |a_{n+1}| < |a_n| r \qquad \text{whenever} \qquad n \geq N$$

So, $|a_{N+1}| < |a_N| r \quad |a_{N+2}| < |a_N| r^2, \cdots, |a_{N+s}| < |a_N| r^s$.

$$\text{Hence,} \quad \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} a_n + \sum_{s=1}^{\infty} |a_{N+s}| < \sum_{n=1}^{N} a_n + |a_N| \sum_{s=1}^{\infty} r^s$$

Example

The series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ is convergent because

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \\ &= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1 \end{aligned}$$

Example

The series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent because

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \\ &= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \\ &= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \to e > 1 \end{aligned}$$

The Root Test

Theorem

- *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*
- *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ (resp. if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$), then the Ratio Test (resp. the Root Test) is inconclusive.

Example

The series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent because

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

11.7 Strategy for Testing Series

1. Apply the Test for Convergence: $\lim_{n\to\infty} a_n = 0$.
2. If a series is similar to power series $\sum \frac{1}{n^p}$ or geometric series $\sum \frac{1}{p^n}$, try the Comparison Tests.
3. For alternating series try the Alternating Series Test.
4. For series involving factorials and other products try the Ratio Test.
5. For series of the form $(a_n)^n$, try the Root Test.
6. If nothing works, try the Integral Test.
7. Try again harder :)

11.8 Power Series

Power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots +$$

The following series is called power series centered in $x = a$:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots +$$

Example

For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| = \frac{1}{1+1/n}|x-3| \to |x-3|$$

So, the series is convergent if $|x-3| < 1$, i.e. for $2 \leq x < 4$.

Example

For what values of x does the series $\sum_{n=1}^{\infty} n!x^n$ converge? We use the ratio test for $x \neq 0$:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty}(n+1)|x| = \infty$$

Hence, the series is convergent for $x = 0$ only.

Example

For what values of x does the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ converge? We also use the ratio test:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2}\cdot\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\ &= \frac{x^2}{4(n+1)^2} \to 0 < 1\end{aligned}$$

Hence, the series is convergent for all x.

Theorem

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only 3 possibilities:

- *The series converges only for $x = a$*
- *The series converges for all x*
- *There is a number $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$*

In the last case the number R is called **radius of convergence.**

The convergence at points $x = a \pm R$ is not specified by the theorem and must be investigated separately.

The range of x for which the series is convergent is called **interval of convergence.**

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{n\,(x+2)^n}{3^{n+1}}$. We have

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)}\right| \\ &= \left(1+\frac{1}{n}\right)\frac{|x+2|}{3} \to \frac{|x+2|}{3} \end{aligned}$$

By the ratio test, the series converges if $|x+2|/3 < 1$, that is $|x+2| < 3$, and diverges if $|x+2| > 3$. Hence, the radius of convergence $R = 3$.

The inequality $|x+2| < 3$ is equivalent to $-5 < x < 1$. At the endpoints $x = -5$ and $x = 1$ the series becomes

$$\sum_{n=0}^{\infty} \frac{n\,(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n n \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{n\,3^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n$$

In either case the series is divergent at the endpoints.

11.9 Representation of Functions as Power Series

One of fundamental representations is already known to us:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

We can use it in many other cases.

Example

Express $1/(1+x^2)$ as the sum of power series. One has

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n \\ &= \sum_{n=1}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots\end{aligned}$$

The interval of convergence is $|x^2| < 1$, that is, $|x| < 1$.

Example

Express $\frac{1}{2+x}$ as the sum of power series. We have:

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n\end{aligned}$$

The interval of convergence is $|-x/2| < 1$, that is $|x| < 2$.

Example

Express $\frac{x^3}{2+x}$ as the sum of power series. One has:

$$\begin{aligned}\frac{x^3}{2+x} &= x^3 \cdot \frac{1}{x+2} = x^3 \cdot \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n = \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{2^{n-2}}x^n \\ &= \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots\end{aligned}$$

Differentiation and Integration of Power Series

Theorem

In the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$ then the function defined by

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (hence, continuous) on $(a-R, a+R)$ and

- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
- $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the powers series are both R.

In other terms, under the conditions of the Theorem,

$$\begin{aligned}
\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] &= \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n(x-a)^n\right] \\
\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx &= \sum_{n=0}^{\infty} \int c_n(x-a)^n \, dx
\end{aligned}$$

Example

Express $1/(1-x)^2$ as the sum of power series. We have:

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots = \sum_{n=1}^{\infty} x^n \\
\frac{1}{(1-x)^2} &= \left(\frac{1}{1-x}\right)' = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}
\end{aligned}$$

The radius of convergence is $R = 1$.

Example

Find a power series representation for $\ln(1+x)$. One has:

$$
\begin{aligned}
\ln(1+x) &= \int \frac{dx}{1+x} = \int (1 - x + x^2 - x^3 + \cdots)\, dx \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + C
\end{aligned}
$$

To find C we put $x = 0$ into the above equation:
$\ln(1+0) = C = 0$. Hence,

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

The radius of convergence is $R = 1$.

Example

Find a power series representation for $\tan^{-1} x$. One has:

$$\begin{aligned} \tan^{-1} x &= \int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 - x^6 + \cdots)\, dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{6} + \cdots \end{aligned}$$

To determine C we plug in $x = 0$: $C = \tan^{-1}(0) = 0$. Therefore,

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

The radius of convergence is $R = 1$.

Example

Evaluate $\int [1/(1+x^7)]\, dx$ as the sum of power series. Approximate $\int_0^{0.5} [1/(1+x^7)]\, dx$ correct to within 10^{-7}.

$$
\begin{aligned}
\frac{1}{1+x^7} &= \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n \\
&= \sum_{n=0}^{\infty} (-1)^n x^{7n} = 1 - x^7 + x^{14} + \cdots \\
\int \frac{dx}{1+x^7} &= \int \sum_{n=0}^{\infty} (-1)^n x^{7n}\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} \\
&= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} + \frac{x^{22}}{22} + \cdots \\
\int_0^{0.5} \frac{dx}{1+x^7} &= \left[x - \frac{x^8}{8} + \frac{x^{15}}{15} + \frac{x^{22}}{22} + \cdots \right]_0^{1/2}
\end{aligned}
$$

Therefore,

$$\int_0^{0.5} \frac{dx}{1+x^7} = \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} + \cdots$$

If we take the first $n = 3$ terms of this alternating sum, the approximation error will be less than $a_4 = \frac{1}{29 \cdot 2^{29}} \approx 6.4 \cdot 10^{-11}$.

Finally, we get

$$\int_0^{0.5} \frac{dx}{1+x^7} \approx \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374$$

11.10 Taylor and Maclaurin Series

Theorem

If a function f has a power series representation at a, i.e.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

In other terms, Taylor series of f at a is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

If $a = 0$ then Taylor series becomes Maclaurin series.

Proof.

By setting $x = a$ in the power series for f we get $c_0 = f(a)$.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Differentiating the last equation we derive

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

By setting $x = a$ we obtain $c_1 = f'(a) = f'(a)/1!$.

$$f''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3c_4(x-a)^2 + \cdots$$

Setting $x = a$ we get $f''(a) = 2c_2$, so $c_2 = f''(a)/2 = f''(a)/2!$.
Similarly,

$$f^{(n)}(x) = n(n-1)(n-2)\cdots 2c_n + (n+1)n(n-1)\cdots 3c_{n+1}(x-a) + \cdots$$

Setting $x = a$ we get

$$f^{(n)}(a) = n!c_n$$

Example

Find Maclaurin series for e^x and its radius of convergence. We get $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$. Hence,

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Since

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \to 0 < 1$$

the series converges for all x. But we are not done yet and need to show that e^x does admit the power series repres. For that we examine the ***n*-th degree Taylor polynomial**

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}}{i!} (x-a)^i$$

and prove that for the **remainder** of the Taylor series $R_n(x) = f(x) - T_n(x)$ it holds: $\lim_{n \to \infty} R_n(x) = 0$.

Theorem

If $f(x) = T_n(x) + R_n(x)$ and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$ then f is equal to the sum of its Taylor series on that interval.

Theorem

If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$ then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for} \qquad |X-a| \le d$$

For $f(x) = e^x$ one has $|f^{(n+1)}(x)| = e^x \le e^d$ for $|x| \le d$. So,

$$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1} \qquad \text{for} \quad |X| \le d$$

Since $\lim_{n\to\infty} x^n/n! = 0$ (follows from convergency of the power series for e^x), we get $R_n(x) \to 0$ as $n \to \infty$.

Example

Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x. We get:

$$\begin{aligned} f(x) &= \sin x & \qquad f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \end{aligned}$$

Therefore, the Maclaurin series becomes

$$\begin{aligned} & f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \\ = \quad & x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

Since $f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$, $|f^{(n+1)}(x)| \le 1$, so $R_n(x) \le |x^{n+1}|/(n+1)! \to 0$ as $n \to \infty$.

Example

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}
\cos x &= \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}
\end{aligned}$$

Example

Find the Maclaurin series for $f(x) = x\cos x$. One has

$$x\cos x = x\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n)!}$$

Example

Find the Taylor series for e^x at $a = 2$. Since $f^{(n)}(2) = e^2$ we get

$$e^x = \sum_{n=1}^{\infty} \frac{f^{(n)}}{n!}(x-2)^n = \sum_{n=1}^{\infty} \frac{e^2}{n!}(x-2)^n$$

Example

Find the Maclaurin series for $f(x) = (1+x)^k$, where k is real.

$$\begin{aligned} f(x) &= (1+k)^k & f(0) &= 1 \\ f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\ f''(x) &= k(k-1)(1+x)^{k-2} & f''(0) &= k(k-1) \\ f'''(x) &= k(k-1)(k-2)(1+x)^{k-3} & f'''(0) &= k(k-1)(k-2) \end{aligned}$$

In general,

$$\begin{aligned} f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} \\ f^{(n)}(0) &= k(k-1)\cdots(k-n+1) \end{aligned}$$

This way we obtain

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

This series is called **binomial series**. If k is integer, then the sum is finite. For the convergency one has

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{k(k-1)\cdots(k-n+1)(k-n)x^{n+1}}{(n+1)!} \cdot \right. \\
&\quad \left. \frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right| \\
&= \frac{|k-n|}{n+1}|x| = \frac{\left|1-\frac{k}{n}\right|}{1+\frac{k}{n}}|x| \to |x| \quad \text{as } n \to \infty
\end{aligned}$$

Hence, the binomial series converges for $|x| < 1$ and diverges for $|x| > 1$

Traditional notation for binomial coefficients is

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$$

This way we established the Binomial Series Theorem:

Theorem

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

The binomial series converges at $x = 1$ if $-1 < k \leq 0$ and at $x = \pm 1$ if $k \geq 0$. If k is a positive integer, then the sum is finite, hence always convergent.

Example

Find the Maclaurin series for the function $f(x) = \frac{1}{\sqrt{4-x}}$. We rewrite the function as follows:

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}} = \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

One has:

$$\begin{aligned}
\frac{1}{\sqrt{4-x}} &= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^{n} \\
&= \frac{1}{2}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^{2}+\cdots\right. \\
&\quad \left.+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}\left(-\frac{x}{4}\right)^{n}\right]
\end{aligned}$$

For the convergence it must hold $|-x/4| < 1$, so $R = 4$.

Example

Evaluate $\int e^{-x^2}\,dx$ as infinite series. One has:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

Integrating term by term we obtain:

$$\begin{aligned} \int e^{-x^2}\,dx &= \int \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots \right) dx \\ &= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots \end{aligned}$$

The series allows to evaluate $\int_0^1 e^{-x^2}\,dx$ correct to 0.001 by taking just its 5 first terms, since for the error term one has: $R_5 = 1/(11 \cdot 5!) < 0.001$.

Example

Evaluate $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$. One could apply l'Hospital's rule, but we can also use series instead:

$$
\begin{aligned}
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \frac{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots\right) - 1 - x}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x^2} \\
&= \lim_{x\to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \cdots\right) \\
&= \frac{1}{2}
\end{aligned}
$$

Multiplication and Division of Power Series

If power series are added, subtracted, multiplied or divided, they behave as polynomials.

Example

Find the first three terms in the Maclaurin series for $e^x \sin x$:

$$\begin{aligned} e^x \sin x &= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right) \\ &= x + x^2 + \frac{1}{3}x^3 + \cdots \end{aligned}$$

Example

Same question as above for $\tan x$:

$$\begin{aligned} \tan x &= \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots} \\ &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \end{aligned}$$

11.11 Applications of Taylor Polynomials

The idea is to replace a function with its Taylor polynomial of n-th degree. The main question is how to choose n to achieve the desired accuracy. The answer to this question is based on estimating the remainder

$$|R_n(x)| = |f(x) - T_n(x)|$$

The general methods are as follows:

- For alternating series $\displaystyle\sum_{n=0}^{\infty}(-1)^{n-1}b_n x^n$ use Alternating Series Estimation Theorem: if $\{|b_n|\} \to 0$ monotonically as $n \to \infty$ then
$$|R_n(x)| \leq b_{n+1}|x-a|^{n+1}$$
- In all cases use Taylor's inequality: if $|f^{(n+1)}(x)| \leq M$ then
$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Example

What is the maximum error by using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

for $|x| \leq 0.3$?

Since the Maclaurin series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

is alternating, we can use the Alternating Series Estimation Theorem. For the approximation error we have

$$|R_n(x)| \leq \frac{|x^7|}{7!} = \frac{|x^7|}{5040}$$

For $|x| \leq 0.3$ the approximation error does not exceed 4.3×10^{-8}. In this case the same estimate can be done by using the Taylor's inequality.

Example

What is the accuracy of approximation of $f(x) = \sqrt[3]{x}$ by its Taylor polynomial of degree 2 at $a = 8$ for $7 \le x \le 8$? One has

$$\begin{aligned} f(x) &= x^{1/3} & f(8) &= 2 \\ f'(x) &= \tfrac{1}{3}x^{-2/3} & f'(8) &= \tfrac{1}{12} \\ f''(x) &= -\tfrac{2}{9}x^{-5/3} & f''(8) &= -\tfrac{1}{144} \\ f'''(x) &= \tfrac{10}{27}x^{-8/3} \end{aligned}$$

So, the second degree Taylor polynomial for $f(x)$ is

$$T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

The Taylor series for $\sqrt[3]{x}$ is not alternating for $x < 8$. But since $f'''(x) \le M = \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$ for $x > 7$ we get

$$|R_3(x)| \le \frac{M}{3!}|x-8|^3 \le \frac{0.0021}{3!} \cdot |7-8|^3 < 0.0004$$

Example

In Einstein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the object at rest and c is the speed of light. The kinetic energy is defined by $K = mc^2 - m_0c^2$. Show that for $v \ll c$ this expression agrees with the classical Newtonian physics: $K = m_0v^2/2$.

We rewrite the formula for energy as follows:

$$K = \frac{m_0}{\sqrt{1 - v^2/c^2}} - m_0c^2 = m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right]$$

Denote $x = -v^2/c^2$ and notice that $|x| < 1$.

$$
\begin{aligned}
(1+x)^{-1/2} &= 1 - \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \cdots \\
&= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots
\end{aligned}
$$

The formula for kinetic energy becomes

$$
\begin{aligned}
K &= m_0c^2\left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) - 1\right] \\
&= m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right) \\
&\approx m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2}\right) = \frac{1}{2}m_0v^2
\end{aligned}
$$

Since $K''(x) = \frac{3}{4}m_0c^2(1+x)^{-5/2}$, for velocities $v \leq 100$m/s and $c = 3 \times 10^8$m/s the approximation error does not exceed $|R_1(x)| \leq \frac{M}{2}x < 4.17 \times 10^{-10}m_0$.