Outline

10.1 Curves defined by parametric equations

Suppose that both x and y are functions of a third parameter t:

$$x = f(t), \qquad y = g(t)$$

As t varies the point $(x, y) = (f(t), g(t))$ traces out a curve, called **parametric curve.**

Example

What curve is represented by the following parametric equations?

$$x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

Obviously it is a circle $x^2 + y^2 = 1$.

What if the range of t would be $0 \leq t \leq 4\pi$?

10.2 Calculus with Parametric Curves

If y and x in a parametric curve equation are functions of t then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Assuming $dx/dt \neq 0$ we get

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0}$$

To compute d^2y/dx^2 replace y with dy/dx:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example

Assume a curve C is defined by $x = t^2$, $y = t^3 - 3t$. Since $y = t(t^2 - 3)$ the curve crosses itself at $t = \pm\sqrt{3}$.

$$\frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

For $t = \pm\sqrt{3}$ the slopes of the tangent lines dy/dx are $\pm 6/(2\sqrt{3}) = \pm\sqrt{3}$, so the tangent lines are

$$y = \sqrt{3}(x - 3) \qquad \text{and} \qquad y = -\sqrt{3}(x - 3)$$

To determine concavity we calculate d^2y/dx^2:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{3}{2}\left(1 + \dfrac{1}{t^2}\right)}{2t} = \frac{3(t^2 + 1)}{4t^3}$$

Thus, C is concave upward for $t > 0$ and downward for $t < 0$.

Areas

Since the area under a curve $y = F(x)$ on $[a, b]$ is $\int_a^b F(x)\,dx$, for a curve defined by parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ we have

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt$$

Example

For the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$, $0 \le \theta \le 2\pi$:

$$\begin{aligned} A &= \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)\,d\theta \\ &= r^2\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)\,d\theta \\ &= r^2\int_0^{2\pi}(1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta))\,d\theta \\ &= 3\pi r^2 \end{aligned}$$

Arc Length

For the length of a curve C $y = F(x)$ on $[a, b]$ we have

$$L = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx$$

If C is defined parametrically with $x = f(t)$, $y = g(t)$ on $[\alpha, \beta]$ with $f'(x) > 0$ we get

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (dy/dx)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \frac{dx}{dt}\, dt \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \end{aligned}$$

This formula is also valid if C cannot be expressed in the form $y = F(x)$. To show this we subdivide the interval $[\alpha, \beta]$ with points $t_1, t_2, \ldots, t_n$ on equal-size subintervals of length Δt.

This way we get points $P_1, P_2, \ldots, P_n$ on C so its length L is

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

By the Mean Value Theorem applied to $f(t)$ on $[t_{i-1}, t_i]$ we have

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1}) = f'(t_i^*)\Delta t$$

Similar equation is also valid for $g(t)$ and some $t_i^{**} \in [t_{i-1}, t_i]$, so

$$\Delta x_i = f'(t_i^*)\Delta t \qquad \Delta y_i = g'(t_i^{**})\Delta t$$

Hence,

$$|P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\,\Delta t$$

So, the length becomes

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\,\Delta t = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt$$

Example

Find the curve length given by equations $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. We have $dx/dt = -\sin t$, $dy/dt = \cos t$, so

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{\sin^2 x + \cos^2 x}\, dx \\ &= \int_0^{2\pi} dt = 2\pi \end{aligned}$$

Example

Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$, $0 \leq \theta \leq 2\pi$. We have

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta \\
&= r\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \\
&= r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta = r\int_0^{2\pi} \sqrt{4\sin^2(\theta/2)}\, d\theta \\
&= 2r\int_0^{2\pi} \sin(\theta/2)\, d\theta = 2r[-2\cos(\theta/2)]_0^{2\pi} \\
&= 2r[2+2] = 8r
\end{aligned}
$$

Surface Area

For a curve $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, with f', g' continuous and $g(t) \geq 0$, rotated about the x-axis, the area of the resulting surface is given by

$$\boxed{S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Example

To compute the surface of sphere we rotate a semicircle $x = r\cos t$, $y = r\sin t$, $0 \leq t \leq \pi$:

$$\begin{aligned} S &= \int_0^\pi 2\pi r \sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\ &= 2\pi r \int_0^\pi \sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt \\ &= 2\pi r^2 \int_0^\pi \sin t\, dt = 4\pi r^2 \end{aligned}$$

10.3 Polar Coordinates

Polar coordinates of a point $P = (x, y)$ is the distance between P and $(0, 0)$ and the angle θ between the ray OP and the x-axis. That is, $P = (r, \theta)$. The angle θ is measured in radians.

We extend the coordinates to the case $r < 0$ as follows:

$$(-r, \theta) = (r, \theta + \pi)$$

The same point can be represented in multiple ways:

$$(r, \theta + 2n\pi) = (-r, \theta + (2n + 1)\pi)$$

Conversion formulas:

$$\begin{array}{lll} \text{Polar to Cartesian :} & x = r\cos\theta & y = r\sin\theta \\ \text{Cartesian to Polar :} & r^2 = x^2 + y^2 & \tan\theta = \frac{y}{x} \end{array}$$

Example

Convert $(2, \pi/3)$ from polar to Cartesian coordinates. One has:

$$
\begin{aligned}
x &= r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1 \\
y &= r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}
\end{aligned}
$$

Example

Convert $(1, -1)$ from Cartesian to polar coordinates. One has

$$
\begin{aligned}
r &= \sqrt{x^2+y^2} = \sqrt{1^2+(-1)^2} = \sqrt{2} \\
\tan\theta &= \frac{y}{x} = -1
\end{aligned}
$$

We get the following possible representations:

$$
(\sqrt{2}, -\frac{\pi}{4}) \qquad (\sqrt{2}, \frac{7\pi}{4})
$$

Polar Curves

The graph of polar equation $r = f(\theta)$ of $F(r, \theta) = 0$ consists of all points whose at least one polar expression (r, θ) satisfies the equation.

Example

Find a Cartesian equation of the curve $r = 2\cos\theta$.

Since $x = r\cos\theta$, $\cos\theta = x/r$. Using polar equation we get $\cos\theta = r/2$, so $r/2 = x/r$, or $2x = r^2 = x^2 + y^2$. So, the curve equation is $x^2 + y^2 - 2x = 0$, or

$$(x-1)^2 + y^2 = 1$$

Symmetry

- If a polar equation is unchanged when θ is replaced with $-\theta$, the curve is symmetric about the x-axis.
- If a polar equation is unchanged when r is replaced with $-r$ or θ is replaced with $\theta + \pi$, the curve is symmetric about the pole.
- If a polar equation is unchanged when θ is replaced with $\pi - \theta$, the curve is symmetric about the y-axis.

Tangents to Polar Curves

Rewrite the parametric equation $r = f(\theta)$ as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

One has

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

For tangent line at the pole when $r = 0$ we get

$$\frac{dy}{dx} = \tan\theta \qquad \text{provided } \frac{dr}{d\theta} \neq 0$$

Example

Find the points on the cardioid $r = 1 + \sin\theta$ where the tangent line is horizontal or vertical.

$$\begin{aligned} x &= r\cos\theta = (1+\sin\theta)\cos\theta = \cos\theta + \frac{1}{2}\sin 2\theta \\ y &= r\sin\theta = (1+\sin\theta)\sin\theta = \sin\theta + \sin^2\theta \end{aligned}$$

and

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos 2\theta}$$

So,

$$\begin{aligned} \frac{dy}{d\theta} = 0, &\qquad \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6} \\ \frac{dx}{d\theta} = 0, &\qquad \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

The case $\theta = \frac{3\pi}{2}$ needs a special treatment.

10.4 Areas and Lengths in Polar Coordinates

Recall the area or a circle sector: $A = \frac{1}{2}r^2\theta$.

Let $\mathcal{R}$ be a polar region bounded by the polar curve $r = f(\theta)$ and rays $\theta = a$ and $\theta = b$ with $0 \leq b - a \leq 2\pi$. We divide it into subintervals of equal width $\Delta\theta$ with endpoints $\theta_1, \ldots, \theta_n$. For the sector bounded with θ_{i-1} and θ_i one has

$$\Delta A_i = \frac{1}{2}[f(\theta_i^*)]^2\Delta\theta \qquad \theta_i^* \in [\theta_{i-1}, \theta_i]$$

So the area of $\mathcal{R}$ can be obtained as

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}[f(\theta_i^*)]^2\Delta\theta = \int_a^b \frac{1}{2}[f(\theta_i^*)]^2\, d\theta = \frac{1}{2}\int_a^b r^2\, d\theta$$

Example

Find the are of one loop of the 4-leaved rose $r = \cos 2\theta$.

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2\, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta \\ &= \int_0^{\pi/4} \cos^2 2\theta\, d\theta = \int_0^{\pi/4} \frac{1}{2}(1+\cos 4\theta)\, d\theta \\ &= \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{\pi}{8} \end{aligned}$$

Area between two polar curves

If the curves are $r = f(\theta)$ and $r = g(\theta)$, $a \le \theta \le b$, the following formula is easy to derive:

$$A = \frac{1}{2}\int_a^b \left([f(\theta)]^2 - [g(\theta)]^2\right)\, d\theta$$

Example

Find the area which is inside the circle $r = 3\sin\theta$ and outside of cardioid $r = 1 + \sin\theta$. The intersecting points are determined by

$$3\sin\theta = 1 + \sin\theta \qquad \Rightarrow \qquad \theta = \frac{\pi}{6}, \ \frac{5\pi}{6}$$

Note that the area is symmetric about the vertical axis.

$$\begin{aligned}
A &= \int_{\pi/6}^{\pi/2} (3\sin\theta)^2 \, d\theta - \int_{\pi/6}^{\pi/2} (1+\sin\theta)^2 \, d\theta \\
&= \int_{\pi/6}^{\pi/2} 9\sin^2\theta \, d\theta - \int_{\pi/6}^{\pi/2} (1 + 2\sin\theta + \sin^2\theta) \, d\theta \\
&= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta) \, d\theta \\
&= \int_{\pi/6}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta) \, d\theta = 3\theta - 2\sin 2\theta + 2\cos\theta]_{\pi/6}^{\pi/2} \\
&= \pi
\end{aligned}$$

Arc Length

To find the arc length of the polar curve $r = f(\theta)$, $a \leq \theta \leq b$, we treat θ as a parameter in parametric curve equations:

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Differentiating them we get

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta - r\cos\theta$$

Therefore,

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\
&+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + t^2\cos^2\theta \\
&= \left(\frac{dr}{d\theta}\right)^2 + r^2
\end{aligned}$$

Hence, the formula for the length becomes

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example

Find the length of the cardioid $r = 1 + \sin\theta$.

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta = \int_0^{2\pi} \frac{\sqrt{2+2\sin\theta}\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}}\, d\theta \\
&= \int_0^{2\pi} \frac{2\cos\theta\ d\theta}{\sqrt{2-2\sin\theta}} = -2\sqrt{2}\int_{-\pi/2}^{\pi/2} \frac{d(1-\sin\theta)}{\sqrt{1-\sin\theta}} \\
&= -2\sqrt{2}\int_{-1}^{1} \frac{d(1-z)}{\sqrt{1-z}} = -2\sqrt{2}\cdot 2\,\sqrt{1-z}\,\Big]_{-1}^{1} = 8
\end{aligned}$$

10.5 Conic Sections

Parabolas

Parabola is a set of points in a plane which are equidistant from a fixed point F (focus) and a fixed line (directrix).

If directrix is $y = -p$ and focus is $(0, p)$, the distance from a point (x, y) on the parabola to F is $\sqrt{x^2 + (y - p)^2}$. The distance from (x, y) to the directrix is $|y + p|$, so the parabola equation becomes

$$\sqrt{x^2 + (y - p)^2} = |y + p|$$

which simplifies to

$$x^2 = 4py$$

Similarly, if directrix is $x = -p$ and focus is $(p, 0)$ the parabola equation is

$$y^2 = 4px$$

Ellipses

An ellipse is the set of points in a plane, the sum of whose distances from two fixed points F_1, F_2 (foci) is a constant. If $F_1 = (-c, 0)$, $F_2 = (c, 0)$, and the constant is $2a$ the ellipse equation becomes

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

which simplifies to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad c^2 = a^2 - b^2, \;\; a \geq b > 0$$

If the foci are of the y-axis, i.e. $F_1 = (0, -c)$, $F_2 = (0, c)$, the ellipse equation becomes

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \qquad c^2 = a^2 - b^2, \;\; a \geq b > 0$$

Hyperbolas

A hyperbola is the set of points in a plane, the difference of whose distances from two fixed points F_1, F_2 (foci) is a constant. If $F_1 = (-c, 0)$, $F_2 = (c, 0)$, and the constant is $2a$ the hyperbola equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad c^2 = a^2 + b^2$$

The x-intercepts $(\pm a, 0)$ are called the vertices of hyperbola, and it has asymptotes $y = \pm(b/a)x$.

If the foci are of the y-axis, i.e. $F_1 = (0, -c)$, $F_2 = (0, c)$, the hyperbola equation becomes

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \qquad c^2 = a^2 + b^2$$

Its vertices are $(0, \pm a)$ and asymptotes are $y = \pm(a/b)x$.

Shifted conics

Shifted conic can be obtained by replacing x and y in its equation with $x - h$ and $y - k$, respectively.

Example

Identify the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find its foci. To accomplish it, we complete the squares:

$$\begin{aligned} 4(y^2 - 2y) - 9(x^2 - 8x) &= 176 \\ 4(y^2 - 2y + 1) - 9(x^2 - 8x + 16) &= 176 + 4 - 144 = 36 \\ 4(y-1)^2 - 9(x-4)^2 &= 36 \\ \frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} &= 1 \end{aligned}$$

Hence, it is a shifted hyperbola with $a = 3$, $b = 2$, $c = \sqrt{13}$ whose (original) foci $(0, \pm\sqrt{13})$ are also shifted accordingly and become $(4, 1 \pm \sqrt{13})$.

10.6 Conic Sections in Polar Coordinates

Theorem

Let

- F *be a fixed point (focus)*
- ℓ *be a fixed line (directrix) in a plane*
- $e > 0$ *be a fixed number (eccentricity).*

The set of all points P *in the plane such that*

$$\frac{|PF|}{|P\ell|} = e$$

is a conic section. The conic is

(*i*) *an ellipse, if* $e < 1$

(*ii*) *a parabola, if* $e = 1$

(*iii*) *a hyperbola, if* $e > 1$

Note that for $e = 1$ then we get the definition of parabola.
Let $F = (0,0)$ and ℓ be the line $x = d$. For $P = (r, \theta)$ one has

$$|PF| = r \qquad |P\ell| = d - r\cos\theta$$

The condition $|PF| = e|P\ell|$ becomes

$$r = e(d - r\cos\theta) \quad \text{or} \quad \sqrt{x^2 + y^2} = e(d - x)$$

By squaring both parts after a little algebra we get

$$\begin{aligned} x^2 + y^2 = e^2(d-x)^2 &= e^2(d^2 - 2dx + x^2) \\ (1-e^2)x^2 + 2de^2x + y^2 &= e^2d^2 \end{aligned}$$

After completing the square we obtain

$$\left(x + \frac{e^2 d}{1-e^2}\right)^2 + \frac{y^2}{1-e^2} = \frac{e^2d^2}{(1-e^2)^2}$$

Proof.

For $e < 1$ we get the ellipse equation of the form

$$\frac{(x-h)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $h = -\frac{e^2 d}{1-e^2}$, $a = \frac{ed}{1-e^2}$, $b = \frac{ed}{\sqrt{1-e^2}}$. The foci are at distance c from the ellipse center, where $c = \sqrt{a^2 - b^2} = \frac{e^2 d}{1-e^2} = -h$ and $e = \frac{c}{a}$.

For $e > 1$ we have $1 - e^2 < 0$, so the equation represents a hyperbola. We could write its equation in the form

$$\frac{(x-h)^2}{a^2} - \frac{y^2}{b^2} = 1$$

and derive $e = \frac{c}{a}$ with $c^2 = a^2 + b^2$. □

Therefore, the curve equation in polar coordinates is

$$r = \frac{ed}{1 + e\cos\theta}$$

For directrix of the form $x = -d$, $y = -d$, or $y = d$ the equation can be obtained by rotating the graph on angles π, $-\pi/2$, or $\pi/2$, respectively. For example. for $y = d$ we get the equation $r = \frac{ed}{1+e\cos(\theta-\pi/2)} = \frac{ed}{1+e\sin\theta}$. Thus, we the theorem:

Theorem

A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic setion of eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

Example

The conic equation $r = \frac{10}{3-2\cos\theta}$ can be rewritten as

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

So, $e = 2/3$ and it is an ellipse. The directrix line is at distance $d = \frac{\frac{10}{3}}{e} = 5$ from the origin, so its equation is $x = -5$.

Example

If we replace the equation from the previous example with

$$r = \frac{10}{3 - 2\cos(\theta - \pi/4)}$$

we get an ellipse rotated on angle $\pi/4$ about one of its foci. The directrix line becomes $y = 5\sqrt{2} - x$.